Tunable superconducting resonators
Subharmonic oscillations and manipulation of microwaves

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Department of Microtechnology and Nanoscience (MC2)
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IDA-MARIA SVENSSON
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Quantum Technology Laboratory
Department of Microtechnology and Nanoscience (MC2)
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
Telephone: +46 (0)31-772 1000

Cover:
A histogram illustrating period tripling subharmonic oscillations with an additional probe signal applied slightly detuned from the measurement frequency.

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In this thesis I present different types of manipulations of microwave fields using tunable superconducting resonators. A resonator is made tunable by adding one or more superconducting quantum interference devices, SQUIDs. The SQUID consists of a superconducting loop with two Josephson junctions and acts as a tunable nonlinear inductor. Modulation of the SQUID nonlinearity can be performed to induce different types of non-trivial oscillator dynamics.

The first project I present is on subharmonic oscillations. Here, the SQUID in a tunable resonator is driven with an external signal at an integer multiple of the frequency $\omega$. When $\omega$ is placed slightly below the first resonator mode, I show generation of radiation at $\omega$, which is known as frequency down-conversion. In my measurements, subharmonic oscillations have been detected from period doubling up to period quintupling. For the specific case of period tripling, theory is developed and I show good agreement between theory and experiments.

The second project of the thesis is on a doubly tunable resonator. Here, I show creation of a superconducting resonator with two independently tunable boundary conditions. The idea with this system is to operate the resonator in a breathing or a translational mode and compare the two. For static dc magnetic flux the performance is very good. As a second step, I perform fast modulation of both SQUIDs to generate radiation. Even though some of the measurement results are promising, I also show some contradicting observations. These indicate that the actual modulation mechanism of the SQUID is not pumping of the magnetic flux in the SQUID loop as intended, but rather direct current driving of the SQUID. However, the system remains a promising idea for microwave manipulation and creation of interesting non-classical states.

As a possible application of the doubly tunable resonator, I present a theory proposal on how to use it for measurements on relativistic effects. By using the tunability of the resonator boundary conditions, the resonator can simulate a space rocket and time dilation could be measured. However, due to the crosstalk problems in the doubly tunable resonator this experiment was never realized.

Finally, I finish the thesis by presenting a design for a tunable microwave coupling. Papers E and F show a high on-off ratio for the coupler and demonstrate how it can be used for storage of microwave signals. Furthermore, the tunable coupler can be used to shape a microwave signal by tuning the strength of the coupling.

**Keywords:** subharmonic oscillations, parametric oscillations, frequency down-conversion, driven nonlinear systems, superconducting resonator, tunable resonator, SQUID, circuit-QED, microwave manipulation
Till farmor för din nyfikenhet som inspirerat mig att alltid vilja veta lite mer och till mormor för att du med din envishet övertygat mig om att allt är möjligt.
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Ida-Maria Svensson, Trollhättan, 28/2-2018
List of publications

This thesis is based on the work contained in the following papers:

**Paper A**

**Paper B**

**Paper C**

**Paper D**

**Paper E**

**Paper F**

Other papers that are outside the scope of this thesis:

**Paper I**

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Thesis
1

Introduction

In 1900, Max Planck studied the spectrum of radiation emitted by a black body in thermal equilibrium. He presented the idea that energy only could be emitted in multiples of elementary units, \( E = \hbar \omega \), where \( \hbar \) is Planck’s constant and \( \omega \) the radiating frequency [9]. These discrete packages, or quantas of energy were later named photons [10]. Since photons are quantas of electromagnetic energy, their propagation is governed by Maxwell equations for electromagnetic waves. However, when Einstein was explaining the photoelectric effect, he used the theory proposed by Planck about discrete energy packages and treated the light as particles to explain the effect [11]. It was found that photons could be treated both as particles and as waves, the particle-wave duality.

Photons are very fast moving objects, and to interact with them, it is convenient to trap them. This can be done in cavities. A simple cavity is formed by two mirrors placed opposite to each other, where the photons bounce back and forth. Within the cavity, the photons can interact with for instance an atom under controlled conditions. The field studying the interaction between light and matter confined in a cavity is called cavity quantum electrodynamics, cavity QED [12].

![Figure 1.1](image1.png)

Figure 1.1: (a) Sketch of a Fabry-Perot cavity that confines optical photons between two mirrors. (b) Sketch of a superconducting transmission line resonator where the microwave photons are confined between two capacitors that act as semi-transparent mirrors.
1. INTRODUCTION

In this thesis, the cavities used are microwave transmission line resonators. A piece of microwave transmission line placed between two capacitors resembles a cavity with two semi-transparent mirrors for the electromagnetic field, see Fig. 1.1, replacing the optical cavities with microwave resonators, optical photons with microwave photons, and natural atoms by artificial atoms. The microwave photons are excitations of the electromagnetic field and form the research field of circuit QED [13–16]. The circuits allow for large flexibility, with properties that can be designed using a wide parameter range. In circuitry acting as atoms, energy levels can be tailor made. These atoms are also known as quantum bits, qubits, the building blocks of a quantum computer.

Circuit QED is a promising approach to build a larger quantum network, such as a quantum computer. In these networks, basic components are needed for storing the quantum information. This is in addition to more complicated circuitry to implement quantum algorithms. In this thesis, I use circuit QED to manipulate microwave fields. My contribution to future quantum networks consists of three different systems. Two of the systems, the subharmonic oscillator and the doubly tunable resonator, rely on frequency down-conversion. In these experiments I apply a modulation signal at a given frequency and detect radiation at subharmonics of this modulation frequency. When operated in the quantum regime, the generated radiation could form interesting non-classical states, potentially useful for continuous variable quantum information processing. The third system is a resonator with a tunable external coupling. By turning off the coupling to a fixed frequency resonator, quantum information can be stored in the resonator. A similar tunable coupler could also be designed between quantum bits.

All three systems rely on one key element, the tunable superconducting resonator [17, 18]. A resonator is made tunable by adding a superconducting quantum interference device, SQUID. The SQUID is essentially a tunable nonlinear inductance that can be used to create nonlinear as well as non-classical effects. It also establishes a coupling between the different resonator modes [19]. This coupling can, for instance, be used to create entanglement between resonator modes [20].

This thesis consists of five chapters which give the background of the appended papers and provide a broader view of our results. In addition, I have tried to include interesting material that is unpublished at the time of writing this thesis.

In chapter 2, I introduce the theoretical framework needed to analyse and understand the measurement results. Here, I introduce the key element of the thesis, the tunable resonator. In this chapter, there are two parts which I explore in detail. One is the derivation of the energy spectra of the tunable resonators. This is essential to understand my measurement results that are consequences of interaction between modes. The other detailed derivation is the theory behind the third order subharmonic oscillations. Although this derivation exists in Paper A, the version included in the thesis is more detailed and easier to follow. Furthermore, I present the equations of parametric pumping for a regular single SQUID tunable resonator as well as the doubly tunable resonator. These equations are useful for understanding the results of Paper C. I briefly review the experimental aspect of measuring the twin paradox in a doubly tunable resonator, the detailed calculations are found in Paper D. Finally, I introduce how a tunable resonator can be used as a tunable coupling for the microwave field, the system treated in Papers E and F.

The technical details of the fabrication and measurements are outlined in chapter 3.
Parts of this chapter are rather technical, but are also of high importance for ensuring the accuracy, reliability and repeatability of measurements. I start by briefly describing the different fabrication techniques used. Then I explain how the samples are installed in a measurement setup. Furthermore, I introduce some measurement techniques and discuss some elements of the circuit design.

Measurement results are presented in chapter 4. Here, I first go through the resonator characterization in more detail than what was done in the papers. Among other things, I show dc tuning measurements on the doubly tunable resonators for different flux pump line designs. Then I describe the interesting results that did not fit in Paper A or B due to space limitations. Here, you find a library of measurements from different samples and flux bias values. There is also a discussion on more technical aspects of the measurement results.

The second part of the SQUID modulation treats parametric pumping at two times the resonance frequency. As a reference I show a regular parametric oscillator. Then I continue by reviewing some complementary measurement results from the doubly tunable resonator that did not fit in Paper B. In the final section of chapter 4, I review the most important findings of the tunable coupling system.

In the last chapter of the thesis, I summarise and discuss the outlook of the different systems. Here, I comment on the weaknesses and strengths of the systems and highlight some possible future routes of the projects.
2

Theory

This chapter is an introduction to all the different theoretical concepts needed to understand the results of the thesis. First, I introduce the basic concepts of superconductivity and the microwave resonator. Some portions of this chapter are expanded over and beyond my master thesis [21]. This introduction leads to the key element of the thesis, the tunable superconducting resonator, which is described in detail in section 2.3. Here I have included the full derivations of the energy spectra for the main resonator types used in the thesis.

Then I go through the theory of different types of microwave manipulation and frequency conversion techniques investigated. Section 2.4 gives the background on different types of fast modulation of the SQUID current or flux. This section covers the derivation of the subharmonic oscillations [1] and the theory for the doubly tunable resonator [3]. Derivations of parametric frequency conversion and the parametric oscillator are not included in this thesis but can be found elsewhere [22–25]. Finally, the section 2.5 gives a short introduction to the tunable coupling circuit.

2.1 Superconductivity

A superconductor is a material where current can flow without resistance. Entering the superconducting regime, many electrical components benefit highly from decreased losses, both for low and high frequency applications [26].

In 1911, H. K. Onnes found the superconducting transition in mercury [27]. Cooling mercury to liquid helium temperature he observed a significant drop in electrical resistivity. The temperature, at which the transition occurs, is called the critical temperature, and at this temperature the electrons start to pair up into Cooper pairs [28]. A Cooper pair consists of two electrons which attract each other as a result of electron-phonon interaction. In a crude picture, this can be explained by the fact that electrons are very light and fast, compared to the heavy metal ions. When a fast electron is passing through a metal ion lattice, the slow but positively charged ions are attracted and move closer together, forming a positively charged region attracting the other electron. The theory explaining
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Electrons are fermions and therefore spin up electrons are only allowed to pair with spin down electrons due to the Pauli principle. Furthermore, when the electrons form Cooper pairs, these pairs become bosons. Therefore, the Cooper pairs can condense into a single quantum state $\Psi = \sqrt{n}e^{i\theta}$, where $\theta$ is the superconducting phase, and $\Psi\Psi^* = n$ the Cooper pair density. The minimum energy required to break up a pair is the superconducting gap energy $2\Delta_{BCS}$. The superconducting gap is a feature of the superconductor density of states, which is formed by two energy bands symmetrically around the Fermi energy. The distance between the bands is $2\Delta_{BCS}$. This energy difference gives the threshold for single particle excitations. Another property of the Cooper pairs is that they cannot transfer heat. This is very useful in experimental setups to thermally isolate between different temperature stages.

The most commonly used superconducting materials are aluminium, niobium, tin and lead. These materials have critical temperatures that ranges from 1.2 K for bulk aluminium up to 9.3 K for niobium. They are all conventional superconductors and described by the BCS theory [29]. It can be noted that some materials with good conductance at room temperature like gold, silver and copper never undergo a superconducting transition. This is because good conductance at room temperature is often connected with a weak electron-phonon coupling, while superconductivity arises from this very interaction. Therefore good superconductors can be relatively bad conductors in their normal state.

While the property of zero resistance is very similar for a superconductor and a perfect conductor, there is another property that distinguish them. Superconductors expel any externally applied magnetic field, and thereby display near-perfect diamagnetism, unlike perfect conductors, see illustration in Fig. 2.1. This phenomenon is known as the Meissner effect [30]. However, if the applied field is stronger than the critical field of the superconductor, all Cooper pairs will be broken and the conductor transitions back to the normal state.

Figure 2.1: (a) In a superconductor, the applied magnetic field is expelled due to the Meissner effect. (b) In a perfect conductor, the applied magnetic field penetrates the conductor.
2.1. SUPERCONDUCTIVITY

\[ \Psi_1 = \sqrt{n_1} e^{i \theta_1}, \quad \Psi_2 = \sqrt{n_2} e^{i \theta_2} \]

Figure 2.2: A superconductor-insulator-superconductor (SIS) junction. The white line indicates the BCS wavefunction, and the fact that the wavefunctions \( \Psi_1 \) and \( \Psi_2 \) overlap indicate that Cooper pairs can tunnel.

2.1.1 The Josephson effect

Two superconducting electrodes placed close to each other can allow electron tunnelling through the potential barrier in between. In 1962, Brian Josephson predicted the Cooper pair tunnelling between two superconducting electrodes [31]. Therefore, the effect is called the Josephson effect and the junction formed by the two electrodes is called a Josephson junction. These junctions are important nonlinear building blocks that are used in circuit QED [15]. There are different types of Josephson junctions but in this thesis the junctions consist of two superconducting electrodes, separated by a thin insulating layer, see Fig. 2.2. In these superconducting junctions the current depends on the phase difference of the wave functions on each side of the junction, \( \theta = \theta_2 - \theta_1 \), as

\[ I = I_c \sin \theta. \tag{2.1} \]

This is known as the DC Josephson relation and \( I_c \) is the junction critical current, i.e. the maximum super current that the junction can support. The superconducting phase difference, \( \theta \), can be related to the voltage over the junction as

\[ \frac{d \theta}{dt} = \frac{2e}{\hbar} V, \tag{2.2} \]

known as the AC Josephson relation.

From the Josephson equations, Eqs. (2.1) and (2.2), it can be seen that a Josephson junction behaves as a nonlinear inductor

\[ V = L_J \frac{dI}{dt}, \]

with the inductance

\[ L_J = \frac{\hbar}{2eI_c \cos \theta}. \tag{2.3} \]

The junction also have an intrinsic capacitance. Approximating the junction as a parallel plate capacitance, it can be estimated

\[ C_J = \frac{\epsilon_0 \epsilon_r A}{t}. \tag{2.4} \]

Here \( \epsilon_0 \) is the permittivity of vacuum, \( \epsilon_r \) the relative permittivity of the insulating layer between the electrodes, \( A \) the junction overlap area and \( t \) the thickness of the insulating layer.
2. THEORY

layer. Energy can be stored in the junction as both inductive and capacitive energy. Depending on the frequency regime, the response is different. For frequencies well below the plasma frequency, \( \omega_p = \frac{1}{\sqrt{LJJC}} \), the response is mainly inductive. In this thesis, the junctions are operated mainly in the inductive regime and the inductive energy, the Josephson energy of a junction, is written

\[
E_J = \frac{\hbar I_c^2}{2e}.
\]

(2.5)

2.1.2 The SQUID

Two Josephson junctions connected in parallel in a superconducting loop is called a Superconducting Quantum Interference Device (SQUID). This device is very sensitive to magnetic flux [26]. From the London equations [32] we have the gradient of the superconducting phase \( \theta \). It depends on the supercurrent, \( \vec{J}_s \) and an electromagnetic vector potential, \( \vec{A} \) as

\[
\nabla\theta = -\Lambda \vec{J}_s - \frac{2e}{\hbar} \vec{A},
\]

with the London parameter \( \Lambda = \frac{m_e}{2ne^2} \) and \( m_e \) is the electron mass. Integrating over a closed loop and choosing a path well inside the conductor gives \( \vec{J}_s = 0 \), since supercurrents are only flowing on the surfaces. The integral can be written

\[
\oint \nabla\theta dl = -\oint \Lambda \vec{J}_s dl - \frac{2e}{\hbar} \oint \vec{A} dl
\]

and rewritten

\[
2\pi n - \theta_1 + \theta_2 = 0 - \frac{2e}{\hbar} \int \nabla \times \vec{A} dS = -\frac{2e}{\hbar} \int \vec{B} dA = -\frac{2e}{\hbar} \Phi,
\]

where \( \Phi \) denotes magnetic flux. Introducing the magnetic flux quantum, \( \Phi_0 = \frac{\hbar}{2e} \), this can be rewritten again as

\[
\theta_1 - \theta_2 = 2\pi n + 2\pi \frac{\Phi}{\Phi_0}.
\]
2.1. SUPERCONDUCTIVITY

The total current flowing in the SQUID is

\[ I = I_{c,1} \sin \theta_1 + I_{c,2} \sin \theta_2 = I_{c,1} \sin \left( \theta + \frac{\Phi}{\Phi_0} + 2\pi n \right) + I_{c,2} \sin \left( \theta - \frac{\Phi}{\Phi_0} + 2\pi n \right), \]

where \( \theta = (\theta_1 + \theta_2)/2 \). This can be rewritten

\[ I = (I_{c,1} + I_{c,2}) \left( \sin \theta \cos \frac{\Phi}{\Phi_0} + \frac{I_{c,1} - I_{c,2}}{I_{c,1} + I_{c,2}} \cos \theta \sin \frac{\Phi}{\Phi_0} \right). \]

If we denote the sum of the critical currents of the two junctions \( I_{c,1} + I_{c,2} = I_c \), and the difference \( I_{c,1} - I_{c,2} = \Delta I_c \) then

\[ \frac{dI}{dt} = I_c \left( \cos \theta \cos \frac{\Phi}{\Phi_0} - \Delta I_c I_c \sin \theta \sin \frac{\Phi}{\Phi_0} \right) \frac{d\theta}{dt}. \]

Using the AC Josephson relation, Eq. (2.2), a definition of the SQUID inductance is found

\[ \frac{dI}{dt} = I_c \left( \cos \theta \cos \frac{\Phi}{\Phi_0} - \Delta I_c \frac{I_c}{I_{c}} \sin \theta \sin \frac{\Phi}{\Phi_0} \right) \frac{2 e}{h} V \equiv \frac{1}{L_{SQ}} V. \]

Explicitly the SQUID inductance is written

\[ L_{SQ} = \frac{L_{SQ,0}}{\cos \theta \cos \frac{\Phi}{\Phi_0} - \Delta I_c I_c \sin \theta \sin \frac{\Phi}{\Phi_0}}, \tag{2.6} \]

where \( L_{SQ,0} = h/2eI_c \). Note how similar the SQUID inductance for a symmetric SQUID, \( \Delta I_c = 0 \), is the single junction inductance, Eq. (2.3). The difference is found in the flux dependence. The SQUID inductance, Eq. (2.6), can be rewritten in terms of the current \( I \) instead of the superconducting phase using the DC Josephson relation, Eq. (2.1), as

\[ L_{SQ} = \frac{L_{SQ,0}}{\sqrt{\cos^2 \frac{\Phi}{\Phi_0} + \left( \frac{\Delta I_c}{I_c} \right)^2 \sin^2 \frac{\Phi}{\Phi_0} - \left( \frac{I_c}{I_c} \right)^2}}, \tag{2.7} \]

which for low signal levels, \( I \ll I_c \), is simplified

\[ L_{SQ} = \frac{L_{SQ,0}}{\sqrt{\cos^2 \frac{\Phi}{\Phi_0} + \left( \frac{\Delta I_c}{I_c} \right)^2 \sin^2 \frac{\Phi}{\Phi_0}}}. \tag{2.8} \]

For a symmetric SQUID, \( I_{c,1} = I_{c,2} \), further simplifying the inductance

\[ L_{SQ} = \frac{L_{SQ,0}}{\cos \frac{\Phi}{\Phi_0}}. \tag{2.9} \]

For room temperature characterization of SQUIDs or junctions we have the expression

\[ I_c = \frac{R_Q}{R_N} \frac{e\Delta_{BCS}}{h}, \]

which relates the normal state resistance of the SQUID (junction) to its critical current [33]. Here \( R_Q = h/4e^2 \) is the superconducting resistance quantum, \( \Delta_{BCS} = 1.76k_BT_c \) the superconducting energy gap of aluminium, \( k_B \) is Boltzmann’s constant, and \( T_c \) is the critical temperature, which for bulk aluminium is 1.2 K.
2. THEORY

2.2 Microwave transmission line resonators

The general picture of a resonator is two mirrors placed opposite of each other forming a cavity that confines light: an optical cavity, a confinement where light bounces back and forth and interfere with itself, called a Fabry-Perot cavity [34]. Instead of optical light and mirrors, we use superconducting circuits and impedance differences to confine microwave signals.

The circuits consist of microwave transmission lines and the size of the components is of the order of a wavelength. Therefore, transmission line theory is used instead of normal circuit theory. In transmission line theory, the voltage and current vary in magnitude and phase over distance. In this thesis the transmission lines are of the coplanar waveguide (CPW) type [35]. The CPW consists of a centre conductor, capacitively coupled to a ground plane via a gap, as shown in Fig. 2.4. Compared to other types of transmission lines, the CPW is convenient to fabricate in small scales at the surface of a chip. In superconducting coplanar waveguides, where the width of the centre conductor is much smaller than the wavelength of the transmitted wave, only a quasi-TEM wave can propagate [36].

The voltage and current signals propagating forward and backwards in a microwave transmission line are written

\[ V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x} \]  
\[ I(x) = I^+ e^{-\gamma x} + I^- e^{\gamma x} \]

where \( \gamma = \alpha + i\beta = \sqrt{(R + i\omega L)(G + i\omega C)} \) is the complex propagation constant and \( R, L, G \) and \( C \) are the resistance, inductance, conductance and capacitance of the transmission line. The losses are represented by the parameter \( \alpha \) and hence for a lossless transmission line \( \gamma = i\beta \). The impedance of the transmission line is given by \( Z = \sqrt{(R + i\omega L)/(G + i\omega C)} \), which in a lossless line simplifies to \( Z = \sqrt{L/C} \).

The properties of a coplanar waveguide transmission line are set by the materials of the circuit and substrate as well as geometry: the centre conductor width, \( w \), and the size of the gap between the centre conductor and the ground plane, \( s \), see illustration in Fig. 2.4. From these dimension and material choices, the geometric inductance and
2.2. MICROWAVE TRANSMISSION LINE RESONATORS

Figure 2.5: (a) Schematic of a transmission line resonator with impedance $Z$, physical length $d$ and electrical length $90^\circ$. The resonator is connected via a coupling capacitor to an environment with the characteristic impedance $Z_0$. (b) The voltage and current distribution in the resonator.

The capacitance per unit length can be calculated \[35-38\]

\[
L_0 = \frac{\mu_0 K(k'_0)}{4 K(k_0)} \tag{2.11a}
\]

\[
C_0 = 4\varepsilon_0\varepsilon_{eff} \frac{K(k_0)}{K(k'_0)}, \tag{2.11b}
\]

where $K$ denotes the complete elliptic integral of the first kind and $k_0 = w/(w + 2s)$, and $k'_0 = \sqrt{1 - k_0^2}$. $\mu_0$ and $\varepsilon_0$ are the vacuum permeability and permittivity respectively, and together they give the speed of light in vacuum $c = 1/\sqrt{\mu_0\varepsilon_0}$. In a coplanar waveguide transmission line, part of the signal is transmitted inside the substrate, therefore the permittivity is increased by a scaling factor, $\varepsilon_{eff}$. The scaling depend on substrate material and geometry. Hence, the speed of light in the coplanar waveguide transmission line is $v = 1/\sqrt{\mu_0\varepsilon_0\varepsilon_{eff}} = 1/\sqrt{L_0 C_0}$.

A microwave resonator consists of a finite length transmission line, see illustration in Fig. 2.5(a). The resonator is defined by its impedance $Z$, its length $d$, and the electrical length. The electrical length defines how many wavelengths that fit inside the resonator boundaries at a specific frequency. Here, one quarter of a period at $\omega_1$ implies that the length of the resonator corresponds to a quarter of the first resonator mode wavelength, a quarter wavelength ($\lambda/4$) resonator. The voltage and current in the resonator are distributed as shown in Fig. 2.5(b). As seen, the voltage has an anti node at the coupling capacitor and a node at the grounded end. The current has a node at the capacitively coupled port and an anti node at the grounded end. The resonator is connected to an environment with impedance $Z_0$ via the coupling capacitor $C_c$. 

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2. THEORY

The resonator stores energy as oscillator standing waves. For these standing waves only certain discrete energy values are allowed, i.e. the set of modes in the energy spectra is discrete [39].

Losses in a resonator can be described by its quality factor, which is defined

\[
Q = \frac{\text{Stored energy}}{\text{Dissipated energy/radian}}.
\]

The total Q-value accounts for both internal losses inside the resonator and external losses due to coupling the resonator to the outside world. The total Q-value can be divided into internal/external losses as

\[
\frac{1}{Q_{\text{tot}}} = \frac{1}{Q_{\text{ext}}} + \frac{1}{Q_{\text{int}}}.
\]  \hspace{1cm} (2.12)

The Q-value relates to the line width of the resonance as

\[
Q_{\text{tot}} = \frac{\omega_r}{2\Gamma},
\]  \hspace{1cm} (2.13)

where \(\omega_r\) is the resonance frequency and \(2\Gamma\) is the total damping of the resonator, i.e. the line width of the resonance. The larger the Q-value, the smaller the line width.

Close to resonance the circuit can be modelled as a lumped element circuit. In Fig. 2.6 an equivalent circuit schematic is presented for a resonator capacitively coupled to a transmission line (Fig. 2.5(a)). The resistance represents the internal losses in the resonator. The capacitively coupled end of the resonator acts as a semi-transparent mirror and the grounded end is a perfectly reflecting mirror. The resonance frequency of the resonator is

\[
\omega_r \approx \frac{1}{\sqrt{L_{\text{tot}}(C_{\text{tot}} + C_c)}},
\]  \hspace{1cm} (2.14)

and the internal and external quality factors can be expressed as [39]

\[
Q_{\text{int}} = \frac{\omega_r RC((\omega_r C_c Z_0)^2 + 1 + C_c/C)}{(\omega_r C_c Z_0)^2 + 1},
\]  \hspace{1cm} (2.15a)

\[
Q_{\text{ext}} = \frac{((\omega_r C_c Z_0)^2 + 1)C + C_c}{Z_0 C_c^2 \omega_r}.
\]  \hspace{1cm} (2.15b)
Using the assumption that the coupling is small $\omega_r C_c Z_0 \ll 1$, this can be rewritten

$$Q_{\text{int}} = \omega_r R (C + C_c),$$

(2.16a)

$$Q_{\text{ext}} = \frac{C + C_c}{\omega_r C_c^2 Z_0},$$

(2.16b)

Comparing incoming and outgoing signals for the resonator an expression for the reflection coefficient can be given as follows [39]

$$S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0},$$

(2.17)

where $Z_L$ is the load impedance of the resonator and $Z_0$ is the characteristic impedance of the environment. To simplify calculations the resonator lumped element model can be modified to the circuit diagram in Fig. 2.7. The coupling capacitance $C_c$ and impedance $Z_0$ have been replaced by equivalent parallel elements $\tilde{Z}_0 = 1/\omega^2 r Z_0 C_c^2$ and $\tilde{C}_c \approx C_c$. For this circuit, the external Q-value is written $Q_{\text{ext}} = \omega r \tilde{Z}_0 (C + C_c)$. With parallel elements, the load admittance of the resonator circuit is

$$Y_L = \frac{1}{Z_L} = \frac{1}{R} + i\omega C + C_c + \frac{1}{i\omega L},$$

which can be rewritten as

$$Y_L = \frac{1}{R} + i\omega (C + C_c) \left(1 - \frac{1}{\omega^2 L (C + C_c)}\right),$$

Inserting (2.14) gives

$$Y_L = \frac{1}{R} + i\omega (C + C_c) \left(1 - \frac{\omega_r^2}{\omega^2}\right),$$

where the last part can be simplified further

$$\left(1 - \frac{\omega_r^2}{\omega^2}\right) = \frac{\omega^2 - \omega_r^2}{\omega^2} = \frac{\omega - \omega_r}{\omega} (\omega + \omega_r) = \frac{\Delta \omega}{\omega} \left(1 + \frac{\omega_r}{\omega}\right) = \frac{\Delta \omega}{\omega} \left(1 + \frac{1}{1 + \frac{\Delta \omega}{\omega}}\right) \approx \frac{2 \Delta \omega}{\omega},$$

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assuming $\Delta \omega \ll \omega_r$. Hence

$$Y_L = \frac{1}{R} + 2i(C + C_c)\Delta \omega = \frac{1}{R} + i\frac{2\Delta \omega Q_{int}}{\omega_r R},$$

using (2.16a). To simplify further calculations it is noticed that (2.17) can be rewritten

$$S_{11} = \frac{\bar{Y}_0 - Y_L}{\bar{Y}_0 + Y_L},$$

where $\bar{Y}_0 = 1/\bar{Z}_0$. When inserting the expression for the load admittance the resulting expression is

$$S_{11} = \frac{1}{\bar{Z}_0} - \frac{1}{R} - i\frac{2\Delta \omega Q_{int}}{\omega_r R} = \frac{\bar{Z}_0 - 1 - i2Q_{int}\Delta \omega}{\bar{Z}_0 + 1 + i2Q_{int}\Delta \omega},$$

and using $\frac{R}{\bar{Z}_0} = \frac{Q_{int}}{Q_{ext}}$, the result is

$$S_{11} = \frac{1}{Q_{ext}} - \frac{1}{Q_{int}} - 2i\frac{\Delta \omega}{\omega_r} \frac{1}{Q_{ext}} + \frac{1}{Q_{int}} + 2i\frac{\Delta \omega}{\omega_r}.$$  \hspace{1cm} \text{(2.18)}

This can also be written in terms of total and external Q-values as

$$S_{11} = \frac{2Q_{tot}/Q_{ext}}{1 + 2iQ_{tot}(\omega/\omega_r - 1)} e^{i\varphi_{ext}} - 1.$$  \hspace{1cm} \text{(2.19)}

Here an extra phase factor is added to the external Q-value, $\varphi_{ext}$, that accounts for impedance mismatches between the resonator and environment. In addition, the full environment that the resonator is connected to can be taken into account by adding some extra background scaling

$$S_{11} = S_{11}^{\text{back}}, S_{11}^{\text{ideal}} = k e^{i(\varphi_{off} + (\omega - \omega_r)\tau_{delay})} \left[ \frac{2Q_{tot}/Q_{ext}}{1 + 2iQ_{tot}(\omega/\omega_r - 1)} e^{i\varphi_{ext}} - 1 \right],$$  \hspace{1cm} \text{(2.20)}

where $k$ is a scaling parameter to adjust the background level, $\varphi_{off}$ is a phase offset, and $\tau_{delay}$ is the electrical delay.

Depending on how the internal and external losses in the resonator compare to each other, the resonator can be in different regimes, see Fig. 2.8. When $Q_{ext} > Q_{int}$ the resonator is undercoupled (green), implying that more energy is lost inside the resonator than in the coupling to other circuitry. If $Q_{ext} < Q_{int}$, more energy is lost through the coupling than inside and the resonator is said to be overcoupled (blue) instead. At the critical point were $Q_{int} = Q_{ext}$, the resonator is critically coupled (red). Depending on applications, different regimes are preferable. However, to maximize the intensity of a general output signal, an overcoupled resonator is preferable.

Internal losses of a superconducting resonator can have many reasons. For a detailed description of these loss mechanisms I recommend the theses of de Graaf and Burnett [40, 41]. The internal quality factor can be divided into contributions from each source of loss

$$\frac{1}{Q_{int}} = \frac{1}{Q_{rad}} + \frac{1}{Q_{TLF}} + \frac{1}{Q_{\mu W}} + \frac{1}{Q_{B}} + \frac{1}{Q_{QP}} + ...$$
2.3. THE TUNABLE RESONATOR

Here $Q_{rad}$ represents radiation losses which are usually very small. $Q_{TLF}$ represents dielectric losses. These are mainly due to interaction with two-level fluctuators that are situated in the substrate-superconductor interfaces [42, 43]. This is usually the main loss mechanism when operating the resonator at low temperature and low signal levels. It has been shown that an increase in temperature can lead to reduced losses due to saturation of two-level fluctuators [44]. Furthermore, the two-level fluctuators can also be saturated by increased signal levels [45]. Other possible loss mechanisms can for instance be mistakes in the microwave engineering that makes the system lossy, $Q_{\mu W}$, losses induced by an applied magnetic field $Q_B$, or quasi particle losses $Q_{QP}$.

2.3 The tunable resonator

The superconducting resonator (Fig. 2.9(a)) is made tunable by adding a SQUID (Fig. 2.9(b)), which acts as a tunable inductance. This means that by controlling the magnetic flux in the SQUID loop, the total inductance of the resonator can be controlled in situ. Effectively, this means that the electric length of the resonator can be tuned, and consequently also its resonance frequency. In Fig. 2.9(c) the voltage profile of the first resonator mode is plotted for the static resonator in (a) and the tunable resonator in (b). As seen the SQUID inductance changes the boundary condition of the electromagnetic field in the tunable resonator (red) compared to the static resonator (blue). This tuning is a tuning of the resonator field boundary condition. This can be seen as an analogy of mechanically moving one of the mirrors confining the light in an optical cavity.

2.3.1 The resonator spectrum

In a resonator, energy can only be stored at certain energies. These energy levels form a spectrum of modes. The spacing between the modes depend on the boundary conditions of the resonator and can be different for different types of resonators. The resonator in

---

Figure 2.8: Illustration of an undercoupled, overcoupled and critically coupled resonator. (a) Magnitude of the reflected signal. Note that the response of the overcoupled and the undercoupled resonators are very similar. (b) Phase of the reflected signal. (c) Polar plot of the real and imaginary part of the reflected signal. Here the undercoupled resonator forms just a small dot at the point $S_{11} = 1$. 

---
Figure 2.9: (a) Schematic of a circuit for a coplanar waveguide microwave resonator, similar to Fig. 2.5(a). (b) Resonator with a SQUID attached in the grounded end, a tunable resonator. (c) Voltage profile for the resonators in (a) and (b). Due to the extra inductance of the SQUID, the effective length of the tunable resonator is longer than that of the static resonator.
2.3. THE TUNABLE RESONATOR

The tunable resonator can be modeled by a distributed network of small capacitive and inductive elements, see Fig. 2.10. The field inside the resonator is characterized by the generalized flux \( \psi(x,t) = \int_{\text{inf}}^{t} V(x,t) \, dx \), which can be evaluated for each node in the distributed network. To model the dynamics of the resonator field, the Lagrangian formalism [46, 47] can be used. The Lagrangian expression represents the difference between capacitive and inductive energy in the circuit. Each element can be treated individually and the different contributions are summed up to the Lagrangian of the full circuit.

The Lagrangian for the different parts of the circuit are written as the following, for a coupling capacitor at position \( x = 0 \)

\[
\mathcal{L}_{Cc} = \frac{C_c}{2} \left( \dot{\psi}(0,t) - V_0(t) \right)^2, \tag{2.21}
\]

for the SQUID

\[
\mathcal{L}_{SQ} = \frac{C_{SQ}}{2} \dot{\psi}^2(x_{SQ},t) + \frac{1}{L_{SQ}(\Phi)} \left( \frac{\Phi_0}{2\pi} \right)^2 \cos \left( \frac{2\pi \psi(x_{SQ},t)}{\Phi_0} \right) 
\]

\[
= \frac{C_{SQ}}{2} \dot{\psi}^2(x_{SQ},t) + \frac{1}{L_{SQ}(\Phi)} \left( \frac{\Phi_0}{2\pi} \right)^2 \left[ 1 - \left( \frac{2\pi}{\Phi_0} \right)^2 \frac{\psi^2(x_{SQ},t)}{2} + \ldots \right] \tag{2.22}
\]

and for the coplanar waveguide transmission line

\[
\mathcal{L}_{\text{res}} = \int_{0}^{x_{SQ}} \left( \frac{C_0}{2} \dot{\psi}^2(x,t) - \frac{[\partial_x \psi(x,t)]^2}{2L_0} \right) \, dx. \tag{2.23}
\]

Then the total Lagrangian is a sum of the different contributions. We extract a small action \( \delta S = \int dt \delta \mathcal{L} \) which should be equal to zero. This requirement leads to the boundary
conditions of the resonator energy spectrum. Different resonator designs result in different boundary conditions. The following subsections will present the resonator types used in this thesis.

The quarter-wavelength resonator

Here I will derive the dispersion relation for the resonator in Fig. 2.9(b). At one end, the resonator is capacitively coupled to the transmission line. While the other end of the resonator is grounded via a SQUID. The boundary conditions for the resonator field at the capacitively coupled end, where \( x = 0 \), is \( \psi_0' = 0 \). In the SQUID terminated end, \( x = d \), the boundary condition is less trivial. To understand this we consider a small action

\[ \delta S = \int dt \left[ \delta L_C + \delta L_{\text{res}} + \delta L_{\text{SQ}} \right] = 0. \]

Since the coupling is static it can be assumed \( \delta L_C = 0 \). The resonator contribution is written

\[ \delta S_{\text{res}} = \int dt \int_0^d dx \left[ C_0 \dot{\psi} \delta \dot{\psi} - \frac{1}{L_0} \psi' \delta \psi' \right], \]

where the inductive part is integrated by part as

\[ - \int_0^d dx \frac{1}{L_0} \psi' \delta \psi' = \frac{1}{L_0} \int_0^d \psi'' \delta \psi dx - \frac{1}{L_0} \left[ \psi' \delta \psi \right]^d_0 \]

\[ = \frac{1}{L_0} \int_0^d \psi'' \delta \psi dx - \frac{1}{L_0} \left( \psi'_d \delta \psi_d - \psi'_0 \delta \psi_0 \right). \]

The last term is zero, since the field is static at the capacitively coupled edge, \( \delta \psi_0 = 0 \).

The capacitive part is written

\[ \int_0^d dx \int dt C_0 \dot{\psi} \delta \dot{\psi} = \int_0^d dx C_0 \left[ - \int dt \ddot{\psi} \delta \psi + \left[ \psi \delta \dot{\psi} \right]^{t_0}_0 \right], \]

where \( \left[ \psi \delta \dot{\psi} \right]^{t_0}_0 = 0 \), under the assumption that the state of the system is fixed (\( \delta \psi = 0 \)) at the beginning and the end of time. Then the full resonator contribution to the dynamics is

\[ \delta S_{\text{res}} = - \int dt \left[ C_0 \int_0^d \left( \ddot{\psi} \delta \psi - \frac{1}{v^2} \psi'' \delta \psi \right) dx + \frac{1}{L_0} \psi'_d \delta \psi_d \right] = - \int dt \frac{1}{L_0} \psi'_d \dot{\psi}_d, \quad (2.24) \]

where the wave equation \( \ddot{\psi} - \frac{1}{v^2} \psi'' = 0 \) is used, remember that the phase velocity \( v = 1/\sqrt{L_0 C_0} \). The SQUID contribution is written

\[ \delta S_{\text{SQ}} = \int dt \left[ C_{\text{SQ}} \dot{\psi}_d \delta \dot{\psi}_d - \frac{1}{|L_{\text{SQ}}(\Phi)|} \psi_d \delta \psi_d \right] \quad (2.25) \]

and using the same technique as for the resonator part this results in

\[ \delta S_{\text{SQ}} = - \int dt \left[ C_{\text{SQ}} \ddot{\psi}_d \delta \psi_d + \frac{1}{|L_{\text{SQ}}(\Phi)|} \psi_d \delta \psi_d \right]. \quad (2.26) \]
To sum up
\[ \delta S = \delta S_{\text{res}} + \delta S_{\text{SQ}} = -\int dt \left[ \frac{1}{L_0} \psi_d' \delta \psi_d + C_{\text{SQ}} \ddot{\psi}_d \delta \psi_d + \frac{1}{|L_{\text{SQ}}(\Phi)|} \psi_d \delta \psi_d \right] = 0, \]
which results in the boundary condition for the SQUID terminated end of the resonator
\[ C_{\text{SQ}} \ddot{\psi}_d + \frac{\left| \cos \left( \frac{\Phi_{\text{ext}} \pi}{\Phi_0} \right) \right|}{L_{\text{SQ},0}} \psi_d + \frac{1}{L_0} \psi_d' = 0 \quad (2.27) \]
For static biasing, \( \Phi_{\text{ext}} = \Phi_{dc} \). The field is assumed to be on the form \( \psi_n \propto e^{\pm i t \omega_n} \cos k_n x \), where \( k_n = \omega_n / v \) is the wave number. Inserted in the boundary condition, Eq. (2.27), the resulting equation is
\[ -C_{\text{SQ}} \omega_n^2 e^{\pm i t \omega_n} \cos k_n d + \frac{\left| \cos \left( \frac{\Phi_{\text{ext}} \pi}{\Phi_0} \right) \right|}{L_{\text{SQ},0}} e^{\pm i t \omega_n} \cos k_n d - \frac{k_n}{L_0} e^{\pm i t \omega_n} \sin k_n d = 0, \]
which can be rewritten
\[ k_n d \tan k_n d = \frac{\left| \cos \left( \frac{\Phi_{dc} \pi}{\Phi_0} \right) \right|}{\gamma_0} - c(k_n d)^2. \quad (2.28) \]
Here the inductive participation ratio, \( \gamma_0 = L_{\text{SQ},0} / (L_0 d) \), is the ratio between the SQUID inductance and the resonator inductance and the capacitive participation ratio \( c = C_{\text{SQ}} / C_0 \) is the ratio between the SQUID capacitance and the resonator capacitance. This dispersion relation is derived in several different earlier works, see for example Ref. [22, 24, 47, 48].

The dispersion relation is a transcendental function, meaning that it can not be solved analytically. Instead, it has to be solved numerically. This has been done as a function of flux for the four lowest modes of a \( \lambda/4 \)-resonator, as shown in Fig. 2.11. The frequency on the vertical axis is normalised to the zero flux frequency of the first resonator mode. Multiples of \( \omega_1(0) \) are indicated by dashed black lines. We find that the spectrum is slightly non-equidistant, with the zero flux frequencies of higher modes being slightly below the odd multiples of the fundamental mode frequency. This is a result of the nonlinearity induced by the SQUID. We can also note that the first mode frequency is tuned all the way to zero when approaching half a flux quantum, while the higher modes reach a finite minimum value. In addition, we find that the higher modes have a smaller frequency spacing between maxima and minima. This can be understood if we look at the dispersion relation, the larger the mode number, \( n \), the larger the capacitive term on the right hand side in Eq. 2.28 is, and therefore the smaller the influence of the inductive term that depends on flux.

The doubly tunable resonator
Next we study the doubly tunable resonator, which is grounded at each end via a SQUID, see Fig. 2.12. In the middle, there is a capacitively coupled probe. The coordinate system is defined so that the capacitive coupling is placed at \( x = 0 \) and the SQUIDs at \( x = \pm d/2 \).
Figure 2.11: The four lowest modes of a $\lambda/4$-resonator. Note how the first mode frequency goes to zero at half flux quantas while the higher modes reach a finite value. For this specific calculation the following parameters have been used, $d = 5.08\, \text{mm}$, $I_c = 1.9\, \mu\text{A}$, $\gamma_0 = 7.7\%$, $C_{SQ} = 86\, \text{fF}$, $c = 11\%$ and $v = 1.2 \cdot 10^8\, \text{m/s}$.

Figure 2.12: Schematic of the doubly tunable resonator used in Paper C. Here the resonator has two SQUIDs, one at each end, i.e. two tunable boundary conditions.
Again the coupling is static and can be neglected. The boundary condition for a SQUID terminated resonator end can be found using Eq. (2.27). Then the boundary conditions for the doubly tunable resonator can be written

\[ C_{SQ,l} \ddot{\psi}_{-d/2} + \frac{|\cos (\Phi_{dc,l}\pi/\Phi_0)|}{L_{SQ,0,l}} \psi_{-d/2} - \frac{1}{L_0} \psi'_{-d/2} = 0 \]  \hspace{1cm} (2.29a)

\[ C_{SQ,r} \ddot{\psi}_{d/2} + \frac{|\cos (\Phi_{dc,r}\pi/\Phi_0)|}{L_{SQ,0,r}} \psi_{d/2} + \frac{1}{L_0} \psi'_{d/2} = 0, \]  \hspace{1cm} (2.29b)

where the subscripts \( l/r \) denote the left and right SQUID. Here it is assumed that the SQUIDs are placed at \( x = -d/2 \) and \( x = d/2 \) respectively. We use an ansatz for the resonator field on the form \( \psi_n \propto e^{\pm i\omega_n t} \cos(k_n x + \beta_n) \). Inserting this in the boundary conditions, Eq. (2.29) give

\[
\begin{align*}
(-\omega_n^2 C_{SQ,l} + \frac{|\cos (\Phi_{dc,l}\pi/\Phi_0)|}{L_{SQ,0,l}}) \cos \left( -\frac{k_n d}{2} + \beta_n \right) + \frac{k_n}{L_0} \sin \left( -\frac{k_n d}{2} + \beta_n \right) &= 0 \\
(-\omega_n^2 C_{SQ,r} + \frac{|\cos (\Phi_{dc,r}\pi/\Phi_0)|}{L_{SQ,0,r}}) \cos \left( \frac{k_n d}{2} + \beta_n \right) - \frac{k_n}{L_0} \sin \left( \frac{k_n d}{2} + \beta_n \right) &= 0.
\end{align*}
\]  \hspace{1cm} (2.30a)

(2.30b)

which can be rewritten

\[
\begin{align*}
\left[ (\gamma_l^{-1} - c_l(k_n d)^2) \cos(k_n d/2) - k_n d \sin(k_n d/2) \right] \cos \beta_n + \\
\left[ (\gamma_l^{-1} - c_l(k_n d)^2) \sin(k_n d/2) + k_n d \cos(k_n d/2) \right] \sin \beta_n &= 0, \hspace{1cm} (2.31a)
\end{align*}

(2.31b)

Here the inductive and the capacitive participation ratios are \( \gamma_i = L_{SQ,0,i}/L_0 \cos \Phi_{dc,i}\pi/\Phi_0 \) and \( c_i = C_{SQ,i}/C_0 d, \) where \( i \in \{l, r\} \). From both equations, expressions for \( \tan \beta_n \) can be extracted

\[
\tan \beta_n = \frac{\gamma_l^{-1} - c_l(k_n d)^2 \cos(k_n d/2) - k_n d \sin(k_n d/2)}{\gamma_l^{-1} - c_l(k_n d)^2 \sin(k_n d/2) + k_n d \cos(k_n d/2)} \]  \hspace{1cm} (2.32a)

\[
\tan \beta_n = -\frac{\gamma_r^{-1} - c_r(k_n d)^2 \cos(k_n d/2) - k_n d \sin(k_n d/2)}{\gamma_r^{-1} - c_r(k_n d)^2 \sin(k_n d/2) + k_n d \cos(k_n d/2)}. \]  \hspace{1cm} (2.32b)

The dispersion relation is found by eliminating \( \tan \beta_n \) from Eq. (2.32). Then we get

\[
k_n d \tan (k_n d) \left[ 1 - \frac{1}{(k_n d)^2} \left( \frac{1}{\gamma_l} - c_l (k_n d)^2 \right) \left( \frac{1}{\gamma_r} - c_r (k_n d)^2 \right) \right] = \frac{1}{\gamma_l} - c_l (k_n d)^2 + \frac{1}{\gamma_r} - c_r (k_n d)^2. \]  \hspace{1cm} (2.33)
Figure 2.13: (a-d) The four lowest modes of the doubly tunable resonator. On the horizontal axis, the magnetic flux bias in the left SQUID and on the vertical axis, the magnetic flux bias in the right SQUID. In colour scale you find the resonance frequency of respective mode scaled by the zero flux frequency of the first mode. (e) Linecuts of the graphs in (a-d), solid lines represent $\Phi_{dc,r} = 0$ and dashed lines $\Phi_{dc,r} = 0.5 \Phi_0$. For this simulation the following parameters have been used: $d = 10.16$ mm, $I_c = 1.47 \mu A$, $\gamma_0 = 5\%$, $C_{SQ} = 86$ fF, $c = 5.5\%$ and $v = 1.2 \cdot 10^8$ m/s.
2.3. THE TUNABLE RESONATOR

With numbers of a typical resonator measured during this thesis, Fig. 2.13 illustrates the four lowest modes given by the dispersion relation. The frequency is scaled by the zero flux frequency of the lowest mode. Panel (a) to (d) shows the dc-tuning of the resonance frequency for the individual modes using the two different SQUIDs. In panel (e) line cuts are extracted from the other graphs, the solid lines represent $\Phi_{dc,r} = 0$ and dashed lines $\Phi_{dc,r} = 0.5 \Phi_0$. Here, it is found that there is a mode close to every integer multiple of the lowest resonator mode. This is in contrast to the $\lambda/4$-resonator where the modes only appear close to odd multiples of the lowest mode frequency (Fig. 2.11).

For a symmetric resonator, $\gamma_l = \gamma_r = \gamma$ and $c_l = c_r = c$, the coefficients in the expressions (2.31) are the same

\[
\begin{align*}
\left[ (\gamma^{-1} - c(k_n d)^2) \cos(k_n d/2) - k_n d \sin(k_n d/2) \right] \cos \beta_n + \\
\left[ (\gamma^{-1} - c(k_n d)^2) \sin(k_n d/2) + k_n d \cos(k_n d/2) \right] \sin \beta_n &= 0.
\end{align*}
\] (2.34)

This forms two subsets of modes, even modes

\[
\begin{align*}
(\gamma^{-1} - c(k_n d)^2) \cos(k_n d/2) - k_n d \sin(k_n d/2) &= 0 \\
\sin \beta_n &= 0, \quad \beta_n = \pm j \pi, \quad j = 0, 1, 2, ...
\end{align*}
\] (2.35)

where the phase field, $\psi_n(x,t) \propto e^{-i\omega_n t}(-1)^j \cos(k_n x)$, is symmetric with respect to the centre of the resonator, and odd modes

\[
\begin{align*}
(\gamma^{-1} - c(k_n d)^2) \sin(k_n d/2) + k_n d \cos(k_n d/2) &= 0 \\
\cos \beta_n &= 0, \quad \beta_n = \pm (2j + 1) \pi/2, \quad j = 0, 1, 2, ...
\end{align*}
\] (2.36)

where the phase field, $\psi_n(x,t) \propto e^{-i\omega_n t}(-1)^{j+1} \sin(k_n x)$, is antisymmetric with respect to the resonator centre. For an illustration of symmetric and antisymmetric modes, see Fig. 2.14, where the phase field for the different modes is plotted for $j = 0$. It can be noted that the phase field and thereby the voltage has maximas in the middle for even modes and is zero for odd modes. This means that the coupling between the resonator and the external transmission line is strong for even modes but zero for odd modes.

The coupling resonator

In order to create the tunable coupling in Paper E and F, we use a tunable $\lambda/2$–resonator. Both ends of the resonator have semi-transparent mirrors, and in the middle a SQUID,
see schematic Fig. 2.15. We assume the SQUID to be placed at position \( x = 0 \), and the length of the resonator to be \( d \), i.e. the coupling capacitors are placed at positions \( x = \pm d/2 \). Then the system can be divided into two parts, left and right resonator and two Lagrangians can be set up as

\[
\mathcal{L}_{\text{left}} = C_0 \int_{-d/2}^{-} dx (\dot{\psi}^2 - v^2 \psi'^2) + \frac{C_{\text{SQ}}}{2} \dot{\psi}_{\text{SQ}}^2 + \frac{\psi_{\text{SQ}}^2}{2L_{\text{SQ}}(\Phi)}
\]

\[
\mathcal{L}_{\text{right}} = C_0 \int_{d/2}^{+} dx (\dot{\psi}^2 - v^2 \psi'^2) - \frac{C_{\text{SQ}}}{2} \dot{\psi}_{\text{SQ}}^2 - \frac{\psi_{\text{SQ}}^2}{2L_{\text{SQ}}(\Phi)}
\]

following definitions of Eqs. (2.23) and (2.22). Here the superconducting phase over the SQUID is \( \psi_{\text{SQ}} = \psi(-0) - \psi(+0) \). Following the techniques used before, the resulting boundary conditions are

\[
\frac{1}{L_0} \psi'_{-0} + C_{\text{SQ}} (\ddot{\psi}_{-0} - \ddot{\psi}_{+0}) + \frac{\psi_{-0} - \psi_{+0}}{L_{\text{SQ}}(\Phi)} = 0
\]

\[
-\frac{1}{L_0} \psi'_{+0} + C_{\text{SQ}} (\ddot{\psi}_{+0} - \ddot{\psi}_{-0}) + \frac{\psi_{+0} - \psi_{-0}}{L_{\text{SQ}}(\Phi)} = 0.
\]

We assume the field is on the form

\[
\psi(x < 0) = a_- \cos \left[ k_n(x + d/2) \right] e^{-i\omega_n t}, \quad \psi(x > 0) = a_+ \cos \left[ k_n(x - d/2) \right] e^{-i\omega_n t}.
\]

Inserting the ansatz in Eq. (2.38) gives

\[
a_- \left( -\frac{k_n}{L_0} \sin \frac{k_n d}{2} - \omega_n^2 C_{\text{SQ}} \cos \frac{k_n d}{2} + \frac{1}{L_{\text{SQ}}(\Phi)} \cos \frac{k_n d}{2} \right) + a_+ \left( \omega_n^2 C_{\text{SQ}} \cos \frac{k_n d}{2} - \frac{1}{L_{\text{SQ}}(\Phi)} \cos \frac{k_n d}{2} \right) = 0
\]

\[
a_- \left( \omega_n^2 C_{\text{SQ}} \cos \frac{k_n d}{2} - \frac{1}{L_{\text{SQ}}(\Phi)} \cos \frac{k_n d}{2} \right) + a_+ \left( -\frac{k_n}{L_0} \sin \frac{k_n d}{2} - \omega_n^2 C_{\text{SQ}} \cos \frac{k_n d}{2} + \frac{1}{L_{\text{SQ}}(\Phi)} \cos \frac{k_n d}{2} \right) = 0.
\]
2.3. THE TUNABLE RESONATOR

By taking the determinant of the system equal to zero we find

\[
\sin \frac{k_n d}{2} \left( \frac{k_n d \tan \frac{k_n d}{2}}{2} - \frac{2}{\gamma_0} \cos \frac{\Phi_\pi}{\Phi_0} + 2c(k_n d)^2 \right) = 0, \quad (2.40)
\]

which defines two sets of modes. One set that is independent of the SQUID, where \(\sin(k_n d/2) = 0\), and one set that depend on the SQUID, defined by the parenthesis equal to zero. It can be noted that the SQUID-dependent part is very close to the dispersion relation for a \(\lambda/4\)-resonator, compare with Eq. (2.28). A calculation of the resonator spectrum is found in Fig. 2.16.

For the lowest resonator mode, the capacitive contribution is negligible and the dispersion relation can be rewritten

\[
\frac{1}{k_1 d} \cot \frac{k_1 d}{2} = \frac{\gamma_0}{2 \cos \frac{\Phi_\pi}{\Phi_0}}.
\]
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cot k_1d/2 can be expanded around k_1d/2 = \pi/2 which corresponds to a plain resonator without a SQUID, yielding

\[- \frac{1}{k_1d} \left( \frac{k_1d}{2} - \frac{\pi}{2} \right) = \frac{\gamma_0}{2\cos \frac{\Phi_0}{\Phi_0}},\]

which can be simplified

\[\omega_1 = \frac{\omega_0}{1 + \frac{\gamma_0}{\cos \frac{\Phi_0}{\Phi_0}}},\]

Here \(\omega_0\) denotes the bare resonance frequency of the resonator without a SQUID. When we only care about the lowest mode, this simplified formula is enough.

2.3.2 The nonlinearity parameter and bifurcation

The SQUID in the resonator induces a Kerr-type nonlinearity denoted \(\alpha\). For weak excitation, the nonlinear superconducting resonator can be compared to a Duffing oscillator [49]. The nonlinearity in mode \(n\) depends on the inductive participation ratio \(\gamma\) and is expressed [23, 24]

\[\alpha_n = \frac{\hbar \omega_n^2}{2\gamma E_{L,cav} \left[ \cos(k_n d)^2 \gamma \right]^2 M_n(k_n d)^2},\]

for the \(\lambda/4\)-resonator in Fig. 2.9(b). Here \(M_n = 1 + \sin(2k_n d)/(2k_n d) + 2\cos^2(k_n d)\) is the generalized mode mass, and \(E_{L,cav} = (\hbar/2e)^2/L_0 d\) is the inductive energy of the resonator. Note that \(\alpha\) is flux dependent since \(\omega_n, k_n\) and \(\gamma\) all depend on flux.

For the doubly tunable resonator, the nonlinearity parameter is a sum of the two SQUID contributions

\[\alpha_n = \alpha_{n,r} + \alpha_{n,l} = \frac{\hbar \omega_n^2}{2\gamma_r E_{L,cav} \left[ \cos(k_n d + \beta_n)^2 \gamma \right]^2 M_n(k_n d)^2} + \frac{\hbar \omega_n^2}{2\gamma_l E_{L,cav} \left[ \cos(k_n d - \beta_n)^2 \gamma \right]^2 M_n(k_n d)^2},\]

\[= \frac{\hbar \omega_n^2}{2\gamma_r E_{L,cav} M_n(k_n d)^4} \left[ \cos(k_n d + \beta_n)^4 + \cos(k_n d - \beta_n)^4 \right]. \tag{2.41}\]

and the mode mass for the doubly tunable resonator is \(M_n = 1 + \sin(k_n d) \cos(2\beta_n)/(k_n d) + 2c_r \cos^2(k_n d/2 + \beta_n) + 2c_l \cos^2(k_n d/2 - \beta_n)\).

The nonlinear effect in the tunable resonator arises due to the limited critical current in the SQUIDs. When the resonator is loaded with a microwave signal, a current flows through the SQUID. If this current \(I \ll I_c\), the resonator gives a linear response to a probe signal. If this current \(I \gg I_c\), the SQUID loses its superconductivity and does not behave like a SQUID any more. In this case the resonator response should be weak due to the losses in the normal state metal. However, there is also a regime in the middle where the current does not cause saturation but only a nonlinear response to a probe signal.

If the input microwave field is denoted \(B\) and the field built up inside the resonator is called \(A\), the resonator response is [24]

\[|A|^2 = \frac{2\Gamma_{ext}}{\zeta + i\Gamma} |B|^2, \tag{2.42}\]
2.4. NONADIABATIC MODULATION OF A SQUID

The SQUID nonlinearity can be used in different ways. Effectively, this nonlinearity acts as a tunable inductance, Eq. (2.7), which can be modulated using either an external drive signal, $I$ which drives a current through the SQUID (current driving) or by modulating the flux, $\Phi_{ext} = \Phi_{ac}(t) + \Phi_{dc}$ in the SQUID loop (flux pumping). What I here define as current driving is commonly referred to in the literature as just ”driving the SQUID” or ”driving with an external signal”.

Figure 2.17: Bifurcated standard Duffing oscillator for a fixed value of $|B|^2$. The dashed line illustrates the unstable branch, and the solid line the stable.

where $\zeta = \delta - \delta \omega$. Here $\delta = \omega - \omega_r$ is the detuning between the probe signal and the resonance frequency. The microwave fields $A$ and $B$ are normalized such that $|A|^2$ is given in number of photons and $|B|^2$ is in number of photons per second. The nonlinear shift of the resonance frequency is denoted $\delta \omega$ and can be expressed

$$\delta \omega = -\alpha |A|^2.$$  

(2.43)

To analyse the nonlinear behaviour, the response in Eq. (2.42) can be rewritten as

$$|A|^2 = \frac{2\Gamma_{ext}}{\delta^2 + 2\alpha \delta |A|^2 + |A|^4 + \Gamma^2} |B|^2.$$  

This equation has for some regimes several solutions, $A$, for a given input field, $B$, and detuning $\delta$. The resonator response bifurcates, it splits into two branches, see Fig. 2.17. The dashed black line correspond to the exact resonance that is shifted due to the nonlinearity. The two branches are marked as dashed blue line for the unstable solution, and solid blue line for the stable solution.

2.4 Nonadiabatic modulation of a SQUID

The SQUID nonlinearity can be used in different ways. Effectively, this nonlinearity acts as a tunable inductance, Eq. (2.7), which can be modulated using either an external drive signal, $I$ which drives a current through the SQUID (current driving) or by modulating the flux, $\Phi_{ext} = \Phi_{ac}(t) + \Phi_{dc}$ in the SQUID loop (flux pumping). What I here define as current driving is commonly referred to in the literature as just ”driving the SQUID” or ”driving with an external signal”.

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In the case of current driving, an input signal $B$ is applied to the input port. This signal generates a current through the SQUID. As explained by the DC Josephson relation, $I = I_c \sin \theta$, this current drives the superconducting phase difference in the SQUID. By modulating the SQUID nonlinearity I can generate subharmonic oscillations through frequency down-conversion. My definition of subharmonic oscillations is that I modulate the SQUID at a frequency $n\omega$, where $n$ is an integer, and I measure a response at $\omega$, see illustration in Fig. 2.18.

In the case of flux pumping a symmetric SQUID, the pump directly modulates the inductance of the SQUID, thus acting as a parametric pump. This is a direct modulation of the resonator boundary condition, which could be thought of as the analogy of moving a physical mirror \[50, 51\]. At frequencies around 10 GHz (two times the resonance frequency of a 5 GHz resonator), this is equivalent to a mirror that is moved at a speed around 25% of the speed of light. Flux pumping a symmetric SQUID only gives frequency down-conversion if the flux pump is fixed to an even multiple of $\omega$. However, if the SQUID junctions are different then the SQUID is asymmetric. With an asymmetric SQUID frequency down-conversion can occur and subharmonic oscillations can be measured for both even and odd multiples. The asymmetry of the SQUID generate a current when flux pumped, in turn this current drives the SQUID.

I follow theory and notation from Refs. \[23, 24\]. In this thesis both current driving and flux pumping has been used. Current driving has been used to generate subharmonic oscillations through frequency down-conversion (Paper A) while flux pumping has been used for parametric pumping (Paper C) as well as generating subharmonic oscillations (Paper B) and frequency up-conversion. In this thesis I have chosen to go into detail on the theory behind the current driven period tripling subharmonic oscillations and I present the theory for photon generation through flux pumping the doubly tunable resonator.

The concepts presented here are interesting from a fundamental physics point of view because they allow the manipulation of single photons. An understanding and control of the system could allow for applications in a quantum network as sources of nonclassical states, such as multipartite entanglement, squeezing and Schrödinger cat states.

2.4.1 Period tripling subharmonic oscillations

Recently a theoretical study on period-tripling in the quantum regime has been published \[52\]. Searching some years back in time also theoretical studies of three-photon down-conversion in optical systems are found \[53–55\]. In this thesis the subharmonic oscillations
2.4. NONADIABATIC MODULATION OF A SQUID

Figure 2.19: Frequency diagram indicating the first two modes of a λ/4-resonator with positive anharmonicity, 3ω₁ − ω₂ > 0. The arrows indicate a measurement signal, ω and the drive signal at 3ω. The measurement signal is placed at negative detuning δ₁, which results in a positive detuning δ₂.

are treated quasi-classically.

For current driving of a λ/4 resonator at 3ω, the configuration is found in Fig. 2.19. The measurement frequency, ω, is placed slightly below ω₁, at negative detuning δ₁. Then the driving frequency 3ω is placed slightly above ω₂, assuming an anharmonicity of the resonator spectrum as in Fig. 2.11. We assume a scenario with two interacting modes. In principle also the higher modes could contribute, however since they are far away we neglect these presumably small contributions. The equations of motion can be written in the rotating frame of ω₁ and ω₂ as [56, 57]

\[
\begin{align*}
\dot{a}_1 + (\delta_1 + i\Gamma_1 + \alpha_1 a_1^\dagger a_1 + 2\alpha a_2 a_2^\dagger) a_1 + \tilde{\alpha} a_1^2 a_2 &= 0 \quad (2.44a) \\
\dot{a}_2 + (\delta_2 + i\Gamma_2 + \alpha_2 a_2^\dagger a_2 + 2\alpha a_1 a_1^\dagger) a_2 + \frac{\tilde{\alpha}}{3} a_1^3 &= \sqrt{2\Gamma_{2,ext}} B_2. \quad (2.44b)
\end{align*}
\]

where \( \alpha = \sqrt{\alpha_1 \alpha_2} \) and \( \tilde{\alpha} = \sqrt{\alpha_1^3 \alpha_2} \) and \( a_{1/2} \) and \( a_{1/2}^\dagger \) are the annihilation and creation operators of the first and second resonator mode. In the second equation, the external drive signal, \( B_2 \), is found on the right hand side. On the left hand side, in both equations, contributions from both the Kerr and the cross Kerr effect can be found. In Eq. (2.44a), the last term, \( \tilde{\alpha} a_1^2 a_2 \), gives a very special type of interaction rarely observed. The drive signal populates the second mode which in turn becomes a parametric pump on the first mode. In fact, the existence of the second mode strongly enhances the down-conversion effect. In the bottom equation, an \( a_1^3 \)-term can be found, this is the back-action of the subharmonic oscillations, on the second mode.

The equations of motion in Eqs. (2.44) can be derived from the quantum Hamiltonian

\[
\begin{align*}
H/\hbar &= -\sum_{n=1,2} \left( \delta_n a_n^\dagger a_n + \frac{\alpha_n}{2} a_n^\dagger a_n^2 \right) - 2\alpha a_1^\dagger a_1 a_2^\dagger a_2 \\
&\quad - \frac{\tilde{\alpha}}{3} \left( a_{1/2}^3 a_{1/2}^\dagger + a_{1/2}^3 a_{1/2}^\dagger \right) + \sqrt{2\Gamma_{2,ext}} \left( B_{2} a_{2}^\dagger + B_{2}^* a_{2} \right). \quad (2.45)
\end{align*}
\]

This Hamiltonian generates a phase space distribution with four maxima and minima, four states, one in the middle with zero amplitude, and three in a triangle with finite amplitude, see Fig. 2.20.

For solving Eqs. (2.44) it is convenient to use the parametrization

\[
a_1 = r_1 e^{i\phi_1}, \quad a_2 = \frac{r_2}{\beta} e^{i\phi_2}, \quad \beta = \sqrt{\alpha_2/\alpha_1}.
\]
Then it follows from the equations of motion that in the stationary state, the system is described by

\[
\begin{align*}
\frac{\delta_1}{\alpha_1} + i \frac{\Gamma_1}{\alpha_1} + r_1^2 + 2r_2^2 & \quad r_1 + r_1^2 r_2 e^{-i\theta} = 0 \\
\frac{\delta_2}{\alpha_1} + i \frac{\Gamma_2}{\alpha_1} + \beta^2 (r_2^2 + 2r_1^2) & \quad r_2 + \frac{\beta^2}{3} r_1^3 e^{i\theta} = \frac{\beta}{\alpha_1} \sqrt{2\Gamma_{2,\text{ext}} B_2} e^{-i\phi_2},
\end{align*}
\]

where \( \theta = 3\phi_1 - \phi_2 \). First, note the trivial solution to Eq. (2.46a), \( r_1 = 0 \). This solution describes an oscillator with zero amplitude, which could be called a silent oscillator state. To investigate the stability of this trivial solution we linearise Eq. (2.44a) for a small fluctuation \( \delta a_1 \) around the solution \( a_{1,0} \). This gives the equation

\[
i \delta a_1 + (\delta_1 + i\Gamma_1 + \alpha_1 |a_{1,0} + \delta a_1|^2 + 2\alpha |a_2|^2) (a_{1,0} + \delta a_1) + \bar{\alpha} (a_{1,0}^* + \delta a_1^*) a_2 = 0.
\]

We assume \( a_1 \propto e^{\lambda_0 t} \) and insert the trivial solution \( a_{1,0} = 0 \). We also neglect higher order \( \delta a_1 \)-terms since they are small, yielding \( i\delta a_1 \lambda_0 + (\delta_1 + i\Gamma_1 + 2\alpha |a_2|^2) \delta a_1 = 0 \), which can be rewritten as

\[
\lambda_0 = i(\delta + 2\alpha |a_2|^2) - \Gamma_1.
\]

Since \( \text{Re}[\lambda_0] \) is negative everywhere, the trivial solution is always stable.

Separating Eq. (2.46a) into real and imaginary part, we get

\[
\begin{align*}
\frac{\delta_1}{\alpha_1} + r_1^2 + 2r_2^2 &= -r_1 r_2 \cos \theta \\
\frac{\Gamma_1}{\alpha_1} &= r_1 r_2 \sin \theta.
\end{align*}
\]

Figure 2.20: Phase portrait, contours of the phase space distribution described by the Hamiltonian in Eq. (2.45), of a third order subharmonic oscillator. This specific plot is generated for \( \delta_1/2\pi = 12 \text{ MHz} \), \( \alpha_1/2\pi = 85 \text{ kHz} \) and \( |B_2|^2 = 6.25 \times 10^{10} \text{ photons/s} \).
Then the phase $\theta$ can be eliminated by combining the two equations

$$r_1^2 = -\frac{\delta_1}{\alpha_1} - \frac{3}{2} r_2^2 \pm \sqrt{-\frac{\delta_1}{\alpha_1} r_2^2 - \frac{7}{4} r_2^4 - \frac{\Gamma_1^2}{\alpha_1^2}}.$$  \hspace{1cm} (2.49)

Since $r_1^2$ should be real, this implies the condition $-(\delta_1/\alpha_1)r_2^2 - (7/4)r_2^4 - \Gamma_1^2/\alpha_1^2 > 0$ which defines a range

$$r_2^2 \in \{r_{2-}^2, r_{2+}^2\}, \quad r_{2\pm}^2 = \frac{2|\delta_1|}{7\alpha_1} \left[ 1 \pm \sqrt{1 - \frac{7\Gamma_1^2}{\delta_1^2}} \right], \hspace{1cm} (2.50)$$

where the solution exist. Furthermore, this implies that the possible solutions only exist at finite negative detuning, given by the damping rate

$$\delta_1 \leq -\sqrt{7} \Gamma_1.$$  \hspace{1cm} (2.51)

The phase $\theta = 3\phi_1 - \phi_2$ is given by Eq. (2.48)

$$\sin(3\phi_1 - \phi_2) = \frac{\Gamma_1}{\alpha_1 r_1 r_2} > 0 \hspace{1cm} (2.52a)$$

$$\cos(3\phi_1 - \phi_2) = \pm \sqrt{1 - \left( \frac{\Gamma_1}{\alpha_1 r_1 r_2} \right)^2} = -\frac{\delta_1}{\alpha_1} + r_1^2 + 2r_2^2 \frac{\Gamma_1}{\alpha_1 r_1 r_2}. \hspace{1cm} (2.52b)$$

For every value of $\phi_2$, there are three possible solutions for $\phi_1$, a threefold degeneracy and the solutions are phase shifted by $2\pi/3$ with respect to each other. Summarizing, Eqs. (2.44) have a stable trivial solution, $r_1 = 0$, as well as some non-trivial solutions, $r_1 > 0$ (since it is an amplitude $r_1$ has to be positive). The Eqs. (2.49) and (2.52) give two possible amplitude solutions and a triplet phase solution.

To investigate the stability of the nontrivial solutions we use the linearised system Eq. (2.47). To simplify the analysis we assume $|\delta_1| \gg \Gamma_1$. Remembering that both $r_1$ and $r_2$ go as $\delta_1$, simplifies Eq. (2.48b) to $\sin \theta = 0$, which implies $\theta = 3\phi_1 - \phi_2 = 0, \pi$. In turn, this allows us to simplify Eq. (2.48a) to

$$r_1 \cos \theta = \pm \left( \frac{1}{2} r_2 \pm \sqrt{-\frac{\delta_1}{\alpha_1} - \frac{7}{4} r_2^2} \right), \hspace{1cm} (2.53)$$

where the sign in front of the parentheses is given by the value of the phase $\theta$. We rewrite Eq. (2.47), neglecting the damping and expanding the terms

$$|a_{1,0} + \delta a_1|^2 = (a_{1,0} + \delta a_1)(a_{1,0}^* + \delta a_1^*) = |a_{1,0}|^2 + a_{1,0}\delta a_1^* + a_{1,0}^* \delta a_1 + |\delta a_1|^2 \hspace{1cm} \text{and}$$

$$(a_{1,0}^* + \delta a_1^*)^2 = a_{1,0}^2 + 2a_{1,0}^* \delta a_1 + \delta a_1^2$$

to

$$i\delta a_1 + (\delta_1 + \alpha_1(|a_{1,0}|^2 + a_{1,0}\delta a_1 + a_{1,0}^* \delta a_1) + 2\alpha|a_2|^2)(a_{1,0} + \delta a_1) + \tilde{\alpha}(a_{1,0}^* + 2a_{1,0}^* \delta a_1) a_2 = 0,$$
We are only interested in the dynamic part and therefore we extract this part and we substitute $\delta a_1$ with $\delta a_1 e^{i\phi_1}$. In addition we insert $a_{1,0} = r_1 e^{i\phi_1}$ and $a_2 = \frac{r_2}{\beta} e^{i\phi_2}$, which gives
\[ i\delta \dot{a}_1 + \delta a_1 e^{i\phi_1} (\delta_1 + 2\alpha_1 r_1^2 + 2\alpha_1 r_2^2) + \delta^* a_1 e^{-i\phi_1} (\alpha_1 r_1^2 e^{i2\phi_1} + 2\alpha_1 r_1 r_2 e^{i(\phi_2 - \phi_1)}) = 0. \]

The last term can be rewritten
\[ \delta a_1^* e^{-i\phi_1} (\alpha_1 r_1^2 e^{i2\phi_1} + 2\alpha_1 r_1 r_2 e^{i(\phi_2 - \phi_1)}) = \alpha_1 e^{i\phi_1} \delta a_1(r_1^2 \pm 2r_1 r_2), \]
where the plus is for $\phi_2 - 3\phi_1 = 0$ and the minus sign is for $\phi_2 - 3\phi_1 = -\pi$. Then we have
\[ i\delta \dot{a}_1 + \delta a_1 (\delta_1 + 2\alpha_1 r_1^2 + 2\alpha_1 r_2^2) + \delta^* a_1 (\alpha_1 r_1^2 \pm 2\alpha_1 r_1 r_2) = 0. \]

From Eq. (2.52b) we have $\pm r_1 r_2 = - (\delta_1/\alpha_1 + r_1^2 + 2r_2^2)$, which yields
\[ i\delta \dot{a}_1 + \delta a_1 (\delta_1 + 2\alpha_1 r_1^2 + 2\alpha_1 r_2^2) \mp \delta^* a_1 (2\delta_1 + \alpha_1 r_1^2 + 4\alpha_1 r_2^2) = 0. \]

If we then set $\delta a_1 = x + iy$, assuming $x$ and $y$ are real numbers $\propto e^{\lambda_1 \alpha_1 t}$, and sum up the real and imaginary parts of the equation
\[ \begin{align*}
-\lambda_1 \alpha_1 y + x(-\delta_1 + \alpha_1 r_1^2 - 2\alpha_1 r_2^2) &= 0 \quad (2.54a) \\
\lambda_1 x + 3y(\delta_1 + \alpha_1 r_1^2 + 2\alpha_1 r_2^2) &= 0. \quad (2.54b)
\end{align*} \]

Eliminating $x/y$ gives
\[ -\alpha_1^2 \lambda_1 = 3(\delta_1 + \alpha_1 r_1^2 + 2\alpha_1 r_2^2) (-\delta_1 + \alpha_1 r_1^2 - 2\alpha_1 r_2^2), \]
where $\delta_1$ can be eliminated using Eq. (2.52b),
\[ -\alpha_1^2 \lambda_1 = 3\alpha_1^2 (\mp r_1 r_2) (2r_1^2 \pm r_1 r_2) \]
which in turn can be rewritten
\[ \lambda_1^2 = \pm 6r_1^2 r_2 \left( r_1 \pm \frac{r_2}{2} \right). \quad (2.55) \]

Note that $r_1 \pm \frac{r_2}{2} = \pm \sqrt{-\frac{\delta_1}{\alpha_1} - \frac{7}{4} r_2^2}$ from Eq. (2.53). This yields, for the minus sign in front of Eq. (2.55),
\[ \lambda_1^2 = -6r_1^2 r_2 \left( \pm \sqrt{-\frac{\delta_1}{\alpha_1} - \frac{7}{4} r_2^2} \right). \]
Here $\lambda_1^2 < 0$ for the positive root, i.e. the positive root gives a stable solution. Hence, for a stable solution we need the combination of a minus sign in front of the parentheses and plus in front of the root in Eq. (2.53), which implies that the plus sign in front of the root in Eq. (2.49) corresponds to a stable triplet solution. The other possible sign combination lead to $\lambda^2 > 0$ and hence an unstable solution. To summarize, the stable solution is

$$r_1^2 = \frac{-\delta_1}{\alpha_1} - \frac{3}{2} r_2^2 + \sqrt{-\frac{\delta_1}{\alpha_1} r_2^2 - \frac{7}{4} r_4 - \frac{\Gamma_1^2}{\alpha_1^2}} \tag{2.56}$$

and the phase $\theta = \pi$.

A closer look at Eq. (2.46b) shows it to be of the format of a Duffing oscillator perturbed by the back-action of the subharmonic oscillations. The phase $\phi_2$ is defined by the imaginary part of that equation

$$\frac{\Gamma_2}{\alpha_1} r_2 + \frac{\beta^2}{3} r_1^3 \sin(3\phi_1 - \phi_2) = \frac{\beta}{\alpha_1} \sqrt{2\Gamma_{2,\text{ext}} |B_2|} \sin(\phi_B - \phi_2), \tag{2.57}$$

where $\phi_B$ denotes the phase of the drive signal $B_2$. If we assume small damping, $\Gamma_2 \ll \delta_1$, and remember that $3\phi_1 - \phi_2 = 0, \pi$, we conclude that also $\phi_B - \phi_2 = 0, \pi$. Consequently, given that $3\omega_1 - \omega_2 = \Delta$, the spectrum anharmonicity, the amplitude of the response is given by

$$\left[ \left( \frac{3}{\alpha_1} \frac{\delta_1}{\alpha_1} + \frac{\Delta}{\alpha_1} + \frac{\beta^2}{3} (r_2^2 + 2 r_1^2) \right) r_2 - \frac{\beta^2}{3} r_1^3 \right]^2 + \frac{\Gamma_2^2}{\alpha_1^2} r_2^2 = 2 \beta^2 \frac{\Gamma_{2,\text{ext}}}{\alpha_1^2} |B_2|^2. \tag{2.58}$$

Analytical analysis can only be done on a simplified system. Therefore stability analysis on the full system has to be done graphically. From Eq. (2.58) $r_2^2(\delta)$ can be evaluated for different values of $B_2$, see the contours in Fig. 2.21. Here the phase $\phi_2$ is taken into account as the sign of $B_2$, $B_2 > 0$ corresponds to $\phi_2 = \phi_B$ (red coloured contours) and $B_2 < 0$ corresponds to $\phi_2 = \phi_B + \pi$ (blue coloured contours). The exact resonance $B_2 = 0$ is given by the dashed black line. The analogy with a Duffing oscillator, see Fig. 2.17, tells which solutions are stable and which are unstable. Below resonance the solutions are unstable and above resonance they are stable. This means that in panel (a) of Fig. 2.21 most solutions were $B_2 > 0$ are stable, while the solutions where $B_2 < 0$ are unstable. In panel (b) and (c), the lower branch of $B_2 > 0$ correspond to stable solutions while the blue part is unstable together with the upper red branch. In panel (d), where the anharmonicity has changed sign from positive to negative, there is a second resonance. Above this second resonance the solutions are stable.

To further investigate the stability, line cuts of Fig. 2.21 can be extracted, see Fig. 2.22. Here $B_2$ is squared and no information of $\phi_2$ can be deduced. Unstable branches are marked by dashed lines while the stable are solid lines. In this graph we find, at small $r_2$-values, the lower blue branch from Fig. 2.22, which is unstable. In addition we find a backbending of the curve at larger $r_2$-values most pronounced for smaller anharmonicity $\Delta/\alpha_1 \leq 50$, where $\Delta = 3\omega_1 - \omega_2$. This backbending correspond to the upper red branch in Fig. 2.21, claimed to be unstable. This effectively decreases the upper boundary for the subharmonic oscillations. For $\Delta/\alpha_1 = -10$ there is an additional structure at large $r_2$, this corresponds to the upper blue branch of Fig. 2.21(d) and is supposedly stable.
Figure 2.21: Graphs illustrating the dependence \( r^2_2(\delta) \) given by Eq. (2.58). The sign on the colour scale gives the phase \( \phi_2 \). The curves in the figure can be compared to a Duffing oscillator (Fig. 2.17). Exact resonance, \( B_2 = 0 \), is highlighted by the dashed black line and the branch above the resonance can be assumed stable and below unstable. The four panels are shown for different values of the anharmonicity, \( \Delta = 3\omega_1 - \omega_2 \).
Figure 2.22: Illustration of the $r_2^2(B_2)$-dependence for different anharmonicities, $\Delta = 3\omega_1 - \omega_2$, of the spectrum, i.e. different flux values. The dashed lines correspond to supposedly unstable branches and the solid lines to stable.
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For analysis of measurement data it is convenient to work with simplified formulas. For a resonator where the anharmonicity is much larger than the detuning, $|\delta_1| \ll 3\omega_1 - \omega_2$, and $r_2^2, r_2^2 \lesssim |\delta_1|/\alpha_1$, the expression in Eq. (2.58) can be simplified to

$$(3\omega_1 - \omega_2)^2 r_2^2 = 2\beta^2 \Gamma_{2,\text{ext}} |B_2|^2.$$  \hspace{1cm} (2.59)

Here $r_2^2$ can be rewritten in terms of $r_1^2$ using Eq. (2.49). This is then a direct relation between the drive signal, $B_2$, and the measured output signal, which depend on $r_1^2$. In addition, the maximum output produced by the drive can be calculated by taking the derivative of Eq. (2.49) with respect to $\delta_1$. This gives

$$r_{1,\text{max}}^2 = \frac{4}{7\alpha_1} \left( |\delta_1| + \sqrt{|\delta_1|^2 - 7\Gamma_1^2} \right).$$  \hspace{1cm} (2.60)

Notably the intensity of the oscillations is decreasing when the flux bias approaches $\pm \Phi_0/2$, since the nonlinearity parameter $\alpha_1$ is increasing.

2.4.2 Standard parametric oscillations

Standard parametric oscillations are created by modulating the flux in the SQUID-loop at two times the resonance frequency of a resonator mode or at the sum of two modes, so called degenerate or non-degenerate photon generation [7, 58, 59]. In this section I focus on parametric oscillations in the $\lambda/4$ resonator of Fig. 2.9(b). The dynamics of the flux pumped SQUID in the resonator end is described by the equation of motion (for degenerate pumping)

$$iA_n + \delta A_n + \epsilon A_n^* + \alpha_n |A_n|^2 A_n + i\Gamma_{0,n} A_n = 0.$$  \hspace{1cm} (2.61)

Here, $A_n$ denotes the field inside the resonator in mode $n$, $\delta$ is the detuning between half the pump and the mode frequency, $\delta = \omega_p/2 - \omega_n$. Similar to the subharmonic oscillations the equation of motion contains a term related to the Kerr effect. The effective pump strength, $\epsilon$, is given by the expression

$$\epsilon = \frac{\Phi_{ac}\pi/\Phi_0}{\gamma_0/|\cos (\Phi_{dc}\pi/\Phi_0)|} \frac{\cos^2 k_n d}{M_n(k_n d)^2},$$  \hspace{1cm} (2.62)

where $\Phi_{ac}$ is the flux pump amplitude in number of flux quanta, $\Phi_{dc}$ is the static flux bias, $k_n = \omega_n/v$ is the mode wave number, $\gamma_0$ is the inductive participation defined in Section 2.3.1 and the mode mass $M_n$ is given in Section 2.3.2. The parametric oscillations occur above a certain threshold given by the damping $\Gamma$ and the detuning $\delta$,

$$\epsilon_{\text{thresh. deg}} = \sqrt{\Gamma_n^2 + \delta^2},$$  \hspace{1cm} (2.63)

see the dashed black lines in Fig. 2.23. Note that this is the expression for degenerate pumping, pumping where $\omega_p = 2\omega_n$.

The flux pump modulates the resonance frequency following the tuning curves of Fig. 2.11. These curves are nonlinear, which means that the modulation of the resonance
frequency is nonlinear. The stronger the pump, the more nonlinear is the modulation. This leads to an effective shift of the resonance frequency, a pump induced frequency shift

\[
\omega_n(\epsilon) - \omega_n(0) = -\frac{\chi_n \epsilon^2}{\Gamma_n},
\]

where \(\chi_n = \Gamma_n \cos^2 \left( \frac{\Phi_{dc} \pi}{\Phi_0} \right) / \left( \omega_n \gamma_0 \sin \left( \frac{\Phi_{dc} \pi}{\Phi_0} \right) \right)\). Hence the effective detuning of the pump can be written

\[
\delta_{eff} = \delta + \frac{\chi_n \epsilon^2}{\Gamma_n},
\]

which modifies the threshold condition accordingly

\[
\epsilon_{thresh,eff} = \sqrt{\frac{\Gamma_n^2}{\Gamma_n^2} + \left( \frac{\chi_n \epsilon_{thresh,eff}}{\Gamma_n} \right)^2}.
\]

Solving for \(\epsilon_{thresh,eff}\) gives

\[
\frac{\epsilon_{thresh,eff}}{\Gamma_n} = \frac{1}{\sqrt{2} \chi_n} \sqrt{1 - 2\delta \chi_n \frac{\Gamma_n}{\Gamma_n} \pm \sqrt{1 - 4 \chi_n (\frac{\delta}{\Gamma_n} + \chi_n)}}. \quad (2.64)
\]

This equation is used to calculate the region boundaries (solid lines) presented in Fig. 2.23 for two different flux bias points. The blue lines correspond to the minus sign in Eq. (2.64) and the red line to the plus sign. Within the parametric oscillation region, photons are generated through parametric down-conversion. The number of photons in the resonator is given by the modulus square of the intra resonator field

\[
|A_n|^2 = \frac{1}{\alpha_n} \left( -\delta + \sqrt{\epsilon^2 - \Gamma_n^2} \right). \quad (2.65)
\]

The parametric oscillator can be described by the Hamiltonian

\[
H(Q, P) = \frac{\epsilon - \delta}{2} P^2 - \frac{\epsilon + \delta}{2} Q^2 - \frac{\alpha}{8\hbar} (Q^2 + P^2)^2. \quad (2.66)
\]

This Hamiltonian defines a metapotential, which can be thought of as a potential with potential wells, between which virtual particles can travel. These potential wells define the possible states of the system. The parametric oscillator can be found in three different regimes, as illustrated in Fig. 2.24. In panel (a) the system has a silent state in the middle and two meta-stable π-shifted states on the sides. In the (b) and (c)-panels the middle state has disappeared and there are only the two π-shifted excited states. Finally, in the (d)-panel there is a squeezed state.

For non-degenerate parametric pumping, the pump frequency is close to the sum of two different mode frequencies, \(\omega_p = \omega_n + \omega_m + 2\delta\). The effective pump strength is then

\[
\epsilon = \frac{(\Phi_{ac} \pi / \Phi_0) \tan (\Phi_{dc} \pi / \Phi_0)}{\gamma_0 / |\cos (\Phi_{dc} \pi / \Phi_0)|} \left( \frac{\sqrt{\omega_n \cos (k_n d/2)}}{\sqrt{M_n k_n d}} \right) \left( \frac{\sqrt{\omega_m \cos (k_m d/2)}}{\sqrt{M_m k_m d}} \right). \quad (2.67)
\]
Figure 2.23: Boundaries of the parametric oscillation region. The dashed black lines illustrate the threshold defined by Eq. (2.63). The solid lines represent the threshold taking into account the pump induced frequency shift, Eq. (2.64). Note that the nonlinearity of the pumping is stronger and more pronounced at 0.1 $\Phi_0$ (a) than at 0.3 $\Phi_0$ (b). This can be understood from the fact that the tuning curve is close to linear to a first approximation at larger flux values. The purple dots in (a) correspond to the points for the phase space distributions in Fig. 2.24.

Moreover, the threshold for non-degenerate pumping is

$$\epsilon_{thres,non-deg.}^2 = \Gamma_n \Gamma_m + \delta^2 \left[ 1 - \left( \frac{\Gamma_n - \Gamma_m}{\Gamma_n + \Gamma_m} \right)^2 \right]$$  \hspace{1cm} (2.68)

and the amplitude of the intra resonator field, in mode $n$ is

$$|A_n|^2 = \frac{-\delta + \sqrt{\Gamma_n \Gamma_m} \sqrt{\epsilon^2 - \Gamma_n \Gamma_m}}{\alpha_n + 2\alpha \frac{\Gamma_n}{\Gamma_m}}$$  \hspace{1cm} (2.69)

with an effective nonlinearity parameter $\alpha = \sqrt{\alpha_n \alpha_m}$. Note that the amplitudes of the two modes follow the relation

$$\Gamma_n |A_n|^2 = \Gamma_m |A_m|^2.$$  

### 2.4.3 Flux pumping the doubly tunable resonator

The doubly tunable resonator is in principle two $\lambda/4$-resonators, coupled with a very strong coupling, compare Fig. 2.9(b) and 2.12. The flux pumped double SQUID resonator can also behave as a parametric oscillator. However, the resonator spectrum is different, there are even and odd modes which gives a parity effect. In addition there are two pump tones that can cause interference effects.

In this system, the flux modulated boundary conditions are an analogy of two moving mirrors. Assuming the two pump frequencies are equal, these mirrors can be set to
Figure 2.24: Phase portraits of the different regimes given by the metapotential in Eq. (2.66) of points a-d as indicated by the purple dots in Fig. 2.23. (a) At negative detuning, below the parametric region, the metapotential forms one stable state in the middle and two metastable $\pi$-shifted states on the sides. (b) At the lower boundary, the middle state disappears and there are only two $\pi$-shifted states. (c) Also in this point we find two $\pi$-shifted metastable states. (d) Above the parametric oscillation region the metapotential forms a squeezed state.
oscillate and form a translational mode, shaking resonator, or a breathing mode, where
the effective length of the resonator varies. In literature, several theoretical works from
the 1990s investigate this type of system [60–64]. They predict that if pumping at an even
multiple of the fundamental mode frequency, there will be constructive interference in
the breathing mode and destructive in the translational. On the other hand, if pumping
at an odd multiple, there will be constructive interference in the translational mode and
destructive in the breathing. These predictions had all in mind mechanical motion of the
mirrors, where relativistic speeds are difficult to achieve. However, with a SQUID-tunable
resonator this is not a limiting factor [50, 65].

As already mentioned, the flux pumped doubly tunable resonator is essentially a
parametric oscillator, but with a different mode structure and two pumps instead of one.
The effective pump strength of the system is given by the following set of cases for modes
a and \( b \) being even (e) or odd (o)

\[
a, b \in (e), \quad \epsilon^2 = \frac{\tan^2 \left( \frac{\Phi_{dc}}{\Phi_0} \pi \right)}{4\gamma^2} \left( \frac{\omega_a \cos^2(k_a d/2)}{M_a(k_a d)^2} \right) \left( \frac{\omega_b \cos^2(k_b d/2)}{M_b(k_b d)^2} \right) |\delta f_r + \delta f_l|^2
\]

(2.70a)

\[
a, b \in (o), \quad \epsilon^2 = \frac{\tan^2 \left( \frac{\Phi_{dc}}{\Phi_0} \pi \right)}{4\gamma^2} \left( \frac{\omega_a \sin^2(k_a d/2)}{M_a(k_a d)^2} \right) \left( \frac{\omega_b \sin^2(k_b d/2)}{M_b(k_b d)^2} \right) |\delta f_r + \delta f_l|^2
\]

(2.70b)

\[
a \in (e), b \in (o), \quad \epsilon^2 = \frac{\tan^2 \left( \frac{\Phi_{dc}}{\Phi_0} \pi \right)}{4\gamma^2} \left( \frac{\omega_a \cos^2(k_a d/2)}{M_a(k_a d)^2} \right) \left( \frac{\omega_b \sin^2(k_b d/2)}{M_b(k_b d)^2} \right) |\delta f_r - \delta f_l|^2.
\]

(2.70c)

Even and odd modes of the doubly tunable resonator are defined in Fig. 2.14. Here
\( \delta f_{l/r} = 2\pi \Phi_{ac,l/r} e^{i\phi_{l/r}} / \Phi_0, \ i \in \{l, r\} \), denotes the pump amplitude and phase. The
threshold for the double parametric oscillator is the same as for the regular parametric
oscillator given for degenerate pumping, \( a = b = n \) and \( \omega_l = \omega_r = 2\omega_n \), by Eq. (2.63)
and for nondegenerate pumping, \( a \neq b \) and \( \omega_l = \omega_r = \omega_a + \omega_b \), by Eq. (2.68). We find
in Eqs. (2.70a) and (2.70b) that pumping at the sum of either two even or two odd
modes result in very similar behaviour. Both these give its lowest threshold, constructive
interference, at a phase difference \( \phi_r - \phi_l = 0 \) and a maximum threshold, destructive
interference, for \( \phi_r - \phi_l = \pm \pi \). In contrast, in the case where the pump is the sum of an
even and an odd mode, the interference is the opposite, constructive for \( \phi_r - \phi_l = \pm \pi \) and
destructive for \( \phi_r - \phi_l = 0 \).

In Eq. (2.70) it is assumed that the two SQUIDs have the same flux bias and that
there is no ac crosstalk. If we first allow the two SQUIDs to have different flux bias, the
formulas are written

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \cos^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \cos^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left( \delta f_r + \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \right) \delta f_l \right)^2
\]

(2.71a)

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \sin^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \sin^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left( \delta f_r + \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \right) \delta f_l \right)^2
\]

(2.71b)

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \cos^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \sin^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left( \delta f_r - \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \right) \delta f_l \right)^2
\]

(2.71c)

where the three subequations correspond to the three cases in Eq. (2.70). Then a possible ac crosstalk is inserted using the parameter \( \xi_{ac,i} \), which could have both an amplitude and a phase, \( \xi_{ac,i} = |\xi_{ac,i}| e^{i\phi_{ac,i}} \). This yields

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \cos^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \cos^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left[ \delta f_r + \xi_{ac,l} \delta f_l \right] + \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \left[ \delta f_l + \xi_{ac,r} \delta f_r \right] \right)^2
\]

(2.72a)

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \sin^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \sin^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left[ \delta f_r + \xi_{ac,l} \delta f_l \right] + \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \left[ \delta f_l + \xi_{ac,r} \delta f_r \right] \right)^2
\]

(2.72b)

\[
\epsilon^2 = \frac{1}{4} \left( \omega_a \cos^2 \frac{k_d}{2} \right) \left( \frac{\omega_b \sin^2 \frac{k_d}{2}}{M_a(k_d)M_b(k_d)} \right) \left( \frac{\tan^2 \left( \frac{\Phi_{dc,r}}{\Phi_0} \right)}{\gamma_r^2/\gamma_l^2} \right) \left[ \delta f_r - \xi_{ac,l} \delta f_l \right] - \frac{\tan^2 \left( \frac{\Phi_{dc,l}}{\Phi_0} \right)}{\gamma_l^2/\gamma_l^2} \left[ \delta f_l - \xi_{ac,r} \delta f_r \right] \right)^2
\]

(2.72c)

Of course the analogy of the translational and the breathing mode will be increasingly difficult, if not impossible to make for larger pump crosstalk.

The effective pump strengths of Eqs. (2.71) can be inserted in Eq. (2.64) to extract the parametric regions. If we only look at \( \delta = 0 \), there is no need to include the pump induced frequency shift. For three different types of flux bias points, the parametric thresholds are plotted in Fig. 2.25 for (a) pumping at the sum of two even modes and (b) for pumping at the sum of an even and an odd mode. The horizontal axes indicate the phase difference between the two pump signals \( \phi_r - \phi_l \). The flux bias of the left SQUID is fixt to \( \Phi_{dc,l} = 0.45 \Phi_0 \), which means that it is placed on a steep frequency slope
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2.4.4 Time dilation measurement in a doubly tunable resonator

In the early 20th century Einstein published the theory of relativity [66, 67]. The theory about what happens when things move very fast, at speeds comparable to the speed of light. One effect of relativistic motion is that a moving clock ticks slower than a still standing one, this is called time dilation. Thinking about time dilation leads to the twin paradox. Of two twins, one is sent on a space trip in a rocket, while the other one stays on earth. When the travelling twin returns he is suddenly younger than his twin sister. The paradox is found in the fact that constant motion is relative, from both the travellers and the earthbound twins frame, the other twin should have aged less. The paradox is resolved by taking the acceleration of the travelling twin into account. The acceleration
of the travelling twin breaks the symmetry between the two reference frames.

In Paper D, the translational mode of the doubly tunable resonator system is proposed as a framework for measuring the time dilation of the twin paradox. The idea is that a resonator, loaded with a microwave signal, is sent on a space trip. When it returns after some time, the microwave signal is read out, and compared to a reference signal. Time dilation of the travelling signal would then be manifested by a phase shift. The twins are the observers, Alice and Rob. Alice is staying on earth, i.e. stays static in the laboratory frame, while Rob travels to space. His trip starts and ends at rest with respect to Alice. In Fig. 2.26(a) the trajectories of Alice and Rob’s resonators are plotted in a Minkowski diagram. The Minkowski diagram is a spacetime diagram where the horizontal axis represents space and the vertical axis time. It is constructed such that a photon travelling in vacuum follows a diagonal through the origin of the diagram. Alice’s resonator that stays static on earth does not move on the spacial axis, see the green lines. On the other hand Rob’s resonator boundaries move with respect to the laboratory frame, the red colour corresponds to segments of acceleration and blue corresponds to segments of constant velocity. He travels out in space and back home.

The idea of time dilation measurements in a superconducting resonator is quite simple. By flux tuning of the SQUIDs in the resonator ends, the electromagnetic field boundary conditions, a movement of a mirror is mimicked. There is a limited lifetime of the
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microwave signal in the resonator. This lifetime decreases when the resonator is detuned from zero flux. Hence, there is a limit on the maximum mimicked displacement of the boundary condition that can be reached, typically corresponding to a couple of millimetres for a resonator lifetime of some microseconds. The displacement should be compared with the resonator length that is around 1 cm. This means that Rob can not travel very far and thus the accumulated phase shift becomes very small. However, the trip can be repeated many times, increasing the possible accumulated phase shift. In practice, Rob would be shaking rather than travelling to Mars, illustrated by the trajectory in Fig. 2.26(b). According to calculations in Paper D, phase shifts larger than 100° are achievable with a shaking resonator.

2.4.5 Parametric frequency up-conversion

Frequency up-conversion is in this thesis used for characterisation purposes. It allows for probing the frequency of modes that are outside the bandwidth of the measurement setup. A weak probe tone is applied to an accessible mode around its resonance frequency, $\omega_m$. At the same time the flux in the SQUID loop is modulated at a frequency close to the difference frequency between mode $n$ and mode $m$, $\omega_n - \omega_m$, where $\omega_n > \omega_m$. Then parts of the signal is converted from mode $m$ to mode $n$, which appears as an avoided level crossing, see Fig. 2.27. The resulting frequencies are given by the formulas

$$
\omega_{m,\pm} = \omega_m - \delta \pm \sqrt{\epsilon^2 + \delta^2}
$$
$$
\omega_{n,\pm} = \omega_n + \delta \pm \sqrt{\epsilon^2 + \delta^2},
$$

(2.73)

where $\delta$ is the detuning given by the frequency diagram in Fig. 2.28, and $\epsilon$ is the effective pump strength. Apart from probing modes outside the measurement bandwidth, this could also be used for calibration of the pump strength $\epsilon$.

Frequency up-conversion works well for nonlinear resonators, where the level spacing between the modes is non-equidistant. In the case of an equidistant spectrum the effect also exist, however the pump would address an infinite number of transitions and therefore be very weak for a single transition. This method is also more efficient the closer the

Figure 2.27: Example of frequency up-conversion from mode $n$ to mode $m$. The coupling between the modes is stated as an avoided level crossing.
2.5. THE TUNABLE COUPLING

In order to create larger quantum networks, one important building block is a tunable coupling - an element allowing information transfer to be turned on and off. Paper E and F presents a system layout with two coupled resonators where one of the resonators is frequency-tunable and can be tuned in and out of resonance with the second one, see Fig. 2.29. In this scheme, the tunable resonator controls the coupling between the storage resonator and the outside world. Searching the literature we find several different layouts for a tunable coupling, for instance using a simple tunable element [69] or for architectures using parametric coupling [70–72].

In our experiment, the two resonators are coupled and form a hybrid system, when probed close to resonance they form an avoided level crossing

\[
\omega_\pm = \frac{1}{2}(\omega_1 + \omega_2) \pm \sqrt{g^2 + \left(\frac{\Delta\omega}{2}\right)^2}. \tag{2.74}
\]

Here \(\Delta\omega = \omega_2 - \omega_1\) and \(g = C_c Z_0 (2\omega_1^2 + \omega_2^2)\), the coupling between the resonators, set by the capacitance \(C_c\). The escape rates of the two modes created by the coupled resonances
are given by

\[
\Gamma_{\text{ext}}^{+} = \cos^2\left(\frac{\theta}{2}\right) \Gamma_{\text{ext},1} + \sin^2\left(\frac{\theta}{2}\right) \Gamma_{\text{ext},2},
\]

\[
\Gamma_{\text{ext}}^{-} = \sin^2\left(\frac{\theta}{2}\right) \Gamma_{\text{ext},1} + \cos^2\left(\frac{\theta}{2}\right) \Gamma_{\text{ext},2},
\]

(2.75a) (2.75b)

where \(\theta = \arctan \frac{2g}{\Delta \omega}\). The escape rate of the coupling resonator is

\[
\Gamma_{\text{ext},2} = \frac{\kappa}{2} = \frac{1}{\pi} \omega_2 (Z_0 \omega_2 C_{\text{out}})^2
\]

given directly by the design of the coupling capacitance \(C_{\text{out}}\), while the storage resonator has a residual coupling to the transmission line that is written

\[
\Gamma_{\text{ext},1} = \frac{2}{\pi} \frac{\omega_1 (Z_0 \omega_1 C_C)^2 (Z_0 \omega_1 C_{\text{out}})^2}{1 + (\omega_1 L_{SQ,0}/Z_0)^2}.
\]

The relation between the couplings are given by the parameter \(\xi = \kappa/4g\). This parameter defines three different regimes. First \(\xi < 1\), the underdamped regime. In the underdamped regime, the rate, at which the signal is transferred between the resonators, is larger than the rate, at which it decays to the transmission line. Therefore, the energy has time to oscillate between the resonators before it escapes to the transmission line. For a critically damped system, \(\xi = 1\), the energy can be directly released from the storage resonator to the transmission line without any oscillation. In the overdamped system, \(\xi > 1\), the coupling between the coupling resonator and the transmission line is stronger than the coupling between the storage and the coupling resonator. Hence, the release of a stored signal to the transmission line is limited by a combination of both couplings, but mainly \(C_{\text{out}}\).

Depending on the regime, different types of experiments can be performed using the tunable coupler. In Paper E we present a sample optimised for storage purposes. The storage resonator is loaded with a microwave signal by turning the two resonators on resonance. Thereafter, the coupling resonator is detuned to close the storage, to later be opened by tuning back on resonance and the stored signal can leak out. By choosing the detuning of the resonators, at which the signal is released, the speed and shape of the released signal can be controlled.

In Paper F the focus is, instead of storage of microwave information, on the shaping of a microwave signal. Here, an underdamped system is used. Using a large detuning between the resonators, the coupling resonator can be loaded while the storage resonator stays empty. Then a swap operation can be performed, when the two resonators are tuned on resonance. The microwave signal in the coupling resonator is moved into the empty storage resonator. A second swap operation can be performed to move the signal back to the coupling resonator. The swap protocol allows loading the storage resonator with microwave contents that origin from different frequencies.
For performing the experiments described in the thesis, some engineering techniques are needed. The samples need to be fabricated and connected to a measurement setup. Furthermore, for measurements at milliKelvin temperatures, we need a system that can cool down and stay cold for a longer time. In this chapter I introduce different fabrication methods as well as some cryogenic cooling techniques. I also step through some measurement setups and sample design aspects. This chapter does not contain everything you could know about the different subjects but some of my experiences and learnings.

3.1 Fabrication techniques

The samples have been fabricated in the Nanofabrication Laboratory at Chalmers using standard fabrication methods. In the thesis, different materials and processes have been used and detailed recipes can be found in Appendix A. The substrates used are sapphire and silicon, and the metals niobium and aluminium, with some minor pieces of gold. The following sections present the different fabrication processes.

3.1.1 Substrate pretreatment

For silicon, the most important pretreatment before metal deposition is a dip in Hydrofluoric (HF) acid. The acid removes the native oxide on the surface. However, the oxide is quickly regrown if exposed to air. Therefore, a fast load into vacuum of the deposition system is needed. The silicon oxide hosts two-level fluctuators, and by removing the oxide, losses can be reduced. Further pretreatment can be done by heating the substrate to high temperature, annealing.

The sapphire substrates used in the thesis are c-plane substrates. These are pretreated by annealing at 1050°C in an oxygen/nitrogen atmosphere for some hours. The annealing is done to reduce surface roughness [73].
3. EXPERIMENTAL METHODS

More extensive cleaning of both silicon and sapphire can be done by solvents to remove organic residues. As you can see in the recipes in Appendix A, three of them contain pretreatment with a solvent cleaning and one does not.

3.1.2 Metal deposition

The superconducting circuits consist of metal. During this thesis two different metal deposition techniques have been used, evaporation and sputtering. Both take place in a high vacuum environment.

Evaporation

An electron gun is pointed towards a metal target which evaporates. This vapour then hits the sample, where the metal condenses and forms a thin film. The thickness of the metal is monitored via a vibrating crystal. During evaporation when the metal thickness increases, the crystal vibration frequency changes. A change which through a calibration can be translated to a metal thickness.

Evaporation with an electron gun is possible for various materials but in this thesis it has been used for aluminium, gold, chromium, titanium and palladium.

Sputtering

In sputtering, atoms of a target metal are physically ejected by bombardment of atoms or ions of a transfer gas. The ejected target atoms are then bombarding the substrate with high energy. By controlling the pressure of the transfer gas, the energy contents of the deposition plasma can be controlled. In this thesis, sputtering has mainly been used for deposition of niobium.

3.1.3 Lithography

For lithography we use polymer mixtures called resists. Different resist types are used for different exposure methods. When exposed, the polymer bindings in the resist are changed. Resist exposure can be done using a mask aligner with a UV-light bulb, a laser beam or electrons. The exposed resist is developed in a matching developer. The developer either dissolves the exposed part of the resist (positive resists) or the unexposed part (negative resists) and a pattern is formed.

Photolithography

The photolithography process is illustrated in Fig. 3.1. UV-light is shined through a mask, defining a pattern in the resist. This is a very fast exposure process for large areas, though resolution and alignment precision is usually limited. The limitation depend on the alignment system and the wavelength of the UV-light. Typical alignment accuracy and minimum feature size of photolithography is of the order of 1 \( \mu \text{m} \).
3.1. Fabrication Techniques

(a) (b) (c) Mask
Resist (unexposed/exposed)
Metal film
Substrate

Figure 3.1: Photolithography. A substrate is covered by photoresist, between the resist and the substrate there could be a metal film or the resist could be spun directly on the substrate. (a) Resist exposure by shining UV-light through a mask. (b) The resist changes properties where it was exposed. (c) A developer resolves and removes the exposed resist while leaving the unexposed unaffected. Hence, a pattern has been defined.

Laser lithography

In laser lithography, the pattern is written using a laser beam. The precision and resolution is slightly better than photolithography, but there is no dramatic difference. The advantage with laser writing compared to photolithography is that no mask is needed and for developing designs this is convenient. The disadvantage, compared to photolithography, is that the writing time increases.

Electron beam lithography

As the name suggests, electrons are used for exposure of the resist. This method has the highest resolution, with fine tuned calibration, structures down to a few nm can be exposed. However, this is also the slowest lithography method. Since the electron beam is small, it takes long time for it to scan larger areas.

It can also be noted that the substrate needs proper grounding. For substrates that conduct electricity (at room temperature) such as silicon this is not a problem, but for sapphire, which is an insulating substrate, electron beam exposures can be difficult. The electron beam easily causes charging effects due to trapping of electrons if there is no path to ground. Some various interesting results can be found in Appendix C.2.

3.1.4 Patterning

A pattern is defined in the resist by a suitable lithography method. The resist pattern can be transferred to the metal film and form circuits by two different techniques, etching and lift-off.

Etching

The etching process is illustrated in Fig. 3.2. A substrate is covered with a thin film metal using either evaporation or sputtering, and then coated with a resist layer. The resist is exposed using one of the methods described above, and after development the underlying metal layer is exposed. This metal could then be either physically or chemically etched. In physical etching, also called dry etching, the material is etched by bombardment of ions, which physically remove the metal. In chemical etching, also called wet etching, the
3. EXPERIMENTAL METHODS

substrate is dipped in a solution that dissolves the metal but preferably does not affect the other parts. An advantage of using etching compared to other patterning techniques is that the interface between metal film and substrate can be properly cleaned before metal deposition. A clean metal-substrate interface has been shown to decrease dielectric losses [43, 74]. To even further reduce the dielectric losses, the substrate in open areas can be etched in order to reduce the electromagnetic field inside the substrate [75, 76].

Lift-off

The lift-off process is illustrated in Fig. 3.3. Two resist layers are placed directly on the substrate. The resists are exposed using one of the methods described above and the resist is developed, either using two different developers for the two layers or using a developer that develops the bottom layer more effectively than the top layer. Note how proper development of the two layers cause an undercut as illustrated in Fig. 3.3(d). Metal is deposited, most commonly through evaporation and due to the undercut there is no risk that the metal sticks to the resist walls. Since sputtering generates heat it could damage the resist and is not optimal for lift-off. After metal deposition, the resist is dissolved in a remover and excess metal is lifted off.

In this thesis, the SQUIDs are fabricated using lift-off and two versions of shadow
3.2 Cryo cooling

We do our measurements at low temperatures, typically at tens of mK, to access the superconducting regime of our samples. But also to reduce the amount of thermal noise $k_B T \ll \hbar \omega$, i.e. to have no thermal excitations. These low temperatures are acquired using a cryostat, a cryogenic refrigerator that can reach very low temperatures. There are different types of cryostats; “wet” cryostats, using a bath of $^4$He, and “dry” cryostats, using a pulse tube and a compressor. These two basic cooling techniques give a temperature of $\sim 4\,\text{K}$. To go lower in temperature we use $^3$He. For simple test measurements we can use a so-called $^3$He cryostat. It cools to base temperature using $^3$He evaporation and can reach temperatures around $300\,\text{mK}$. Lower temperatures need a mixture of $^3$He and $^4$He, which is used in a dilution cryostat, allowing for measurements down to $10\,\text{mK}$.

3.2.1 The $^3$He cryostat

A schematic of our $^3$He cryostat can be found in Fig. 3.6. It is a Janis cryostat model He-3-SSV. The inner part of the cryostat, the inner vacuum chamber (IVC) is placed in a $^4$He bath inside a dewar. Liquid $^4$He has a boiling temperature of $4.2\,\text{K}$. Hence, a bath of $^4$He cools the IVC to this temperature. The cryostat then cools in stages with different temperatures and it is of high importance that there is as low thermal contact
3. EXPERIMENTAL METHODS

Figure 3.5: Each panel show both a top view of the structure and a side view. (a) Two layers of resist are exposed and developed. (b) Metal is evaporated at an angle that prevents the perpendicular trench from being filled. (c) Oxidation of the metal. (d) The substrate is rotated $90^\circ$. (e) A second metal layer is evaporated. (f) The resist is removed and a junction is formed.
Figure 3.6: Schematic of our $^3$He-cryostat.
3. EXPERIMENTAL METHODS

between the stages as possible. Therefore, the IVC is kept under vacuum since air would give thermal contact between the outside and the cryostat inside the IVC.

By using the 1K pot, the temperature can be lowered. A pump is turned on that removes the most energetic (warmest) atoms and the pot is refilled, with a rate controlled by the needle valve position, with liquid $^4$He from the bath. The purpose of the 1K-pot is to condense the $^3$He, which can be done when below $\sim 3$K.

The $^3$He is forced out of its storage by heating the sorbtion pump. Then it is condensed in the 1K-pot and the $^3$He-pot is filled with liquid $^3$He. The last step to reach base temperature is to pump on the liquid $^3$He. This is done by cooling the sorbtion pump, liquid $^4$He is taken from the bath through a capillary and evaporates through the flow gauge used to control the flow, i.e. the cooling power.

After a few minutes, the $^3$He-pot reaches the base temperature around 300 mK and cools the cold finger where the sample is mounted. The cryostat then stays cold as long as the $^3$He lasts in the $^3$He-pot. When the $^3$He-pot is empty a new condensation must be performed.

3.2.2 The dilution cryostat

A dilution refrigerator uses a mixture of $^3$He and $^4$He. When this mixture is cooled below 0.7 K a phase separation occurs. The liquid is divided into two phases, one $^3$He-concentrated phase and one $^3$He-poor phase, where the concentrated phase floats on top of the diluted $^3$He-poor phase. By removal of $^3$He from the dilute phase, equilibrium is disturbed, and $^3$He is taken from the concentrated phase to re-establish concentration balance. This process of $^3$He crossing the phase boundary is endothermic, energy is consumed and the temperature is reduced.

The mixture is condensed using either a $^4$He-bath and a 1K-pot, ”wet” cryostat, or a pulse tube cryocooler, the more modern ”dry” cryostat. For a very thorough explanation of the dilution refrigerator, I recommend reading the thesis of Michaël Simoen [22]. The cryostats used in this thesis were mainly two modern ”dry” cryostats (Bluefors LD250). However, also an older ”wet” Oxford cryostat, has been used.

3.3 Measurement setup

The measurement setup is composed of commercially available microwave components. Here, I will mainly focus on the setups that have been used in the dry dilution refrigerators, since this is where the main measurements have been performed. In Fig. 3.7 typical input and output lines are presented. On the input line the cables are thermalised at all the different temperature stages. Attenuators are distributed to minimize heating and noise. The cables are thin coaxes, made of silver plated CuNi, which is a bad thermal conductor. The output line consists of a superconducting coaxial cable from the mixing chamber stage all the way to the amplifier. Superconductors have low loss and very poor thermal conductivity, which make them convenient (but expensive) for usage from low to high temperature stages. The amplifiers are high-electron-mobility-transistor (HEMT) amplifiers from the company Low Noise Factory and developed specially for cryogenic use.
In this thesis, amplifiers with two different frequency ranges have been used, 4 – 8 GHz (LNF-LNC4_8A) and 1 – 12 GHz (LNF-LNC1_12A).

For reflection type measurements, two different setup types have been used, see Fig. 3.8. In (a) the signal is routed through circulators and in (b) through a directional coupler. The signal routing allows the input and output signal to travel different paths, which in turn allows for proper attenuation of the input signal and amplification of the output signal. The microwave components as well as our amplifiers limit the available frequency range for measurements. The circulators are specified only for the 4 – 8 GHz-band and thereby limits the available frequency range for measurements in a circulator setup. The measurement frequency limit in the isolator setup is set by the isolators that are specified in the 3 – 12 GHz-range. However, the directional coupler can be more wideband and allow for a higher frequency drive signal.

3.3.1 Magnetic flux control

Static magnetic flux biasing of a single SQUID can be done by using an external coil, mounted close to the sample. For samples with multiple SQUIDs, the biasing is done on-chip, using on-chip flux tuning lines, to allow for individual control. An on-chip flux tuning line is a transmission line, close to the SQUID, that generates a magnetic field, when a current is applied. These can be designed in many different ways, for some examples see Fig. 4.10. A discussion on flux tuning line designs is found in Section 4.2, in the subsection about the doubly tunable resonator.
3. EXPERIMENTAL METHODS

On-chip flux tuning lines are also used for fast modulation of the magnetic flux. For high frequency modulation, the on-chip flux tuning line is connected to a typical input line (Fig. 3.7) and then grounded either on-chip, on the sample holder or capped by a 50Ω terminator outside the sample box. For a combination of dc and ac signals in the on-chip flux tuning line, a bias-T is used to combine these two at low temperature. If needed, the combined signal can be transported off-chip and a second bias-T can be connected to allow for proper termination of both the ac and dc signal. Typical flux tuning line setups with and without bias-T are sketched in Fig. 3.9.

3.3.2 Sample mounting

The fabricated samples are glued (using Russian BF6-glue) in sample boxes and bonded by aluminium or gold wires, an example is shown in Fig. 3.10. Different types of boxes have been used in the thesis, two designs are shown in Fig. 3.11. Most measurements have been performed using the design in panel (a) and (b), where the bonding is done to a printed circuit board (PCB). However, it was found out that these PCBs cause loss. Therefore my colleges, Andreas Bengtsson and Jonathan Burnett, developed new boxes, panel (c) and (d), where no PCB is used but the sample ground plane is bonded directly to the box and the contact pads are bonded directly to the pins of the SMA-connectors. This design is better from several perspectives, one is the absence of the PCB but also the smaller dimension places possible parasitic box modes at higher frequencies.

3.3.3 Digitizer measurements

We use a digitizer for heterodyne detection of signals, a schematic is sketched in Fig. 3.12. It has an analogue down-conversion step and a digital step that divides the signal into the in- and out-of-phase quadratures. The input signal is down-converted using a mixer. In
3.3. MEASUREMENT SETUP

Figure 3.9: (a) Simple setup for high frequency flux modulation. (b) Using a bias-T both dc and ac signals can be combined in the on-chip flux tuning line. (c) If the on-chip flux tuning line is not grounded on-chip, the signal can be divided back into ac and dc components and terminated properly. Above the 10 mK-stage the ac lines are identical to the input line in Fig. 3.7. The dc lines are twisted pair wires.

Figure 3.10: A sample bonded directly to the copper sample box and the centre pin of the connectors.
3. EXPERIMENTAL METHODS

Figure 3.11: Two types of sample boxes that have been used in this thesis. (a) The outside and (b) inside of an old style sample box using a printed circuit board (PCB). (c) The outside and (d) inside of a newer sample box design. A zoom in on a sample can be found in Fig. 3.10.

Figure 3.12: Schematic of a digitizer. The input signal is down-converted using a mixer. Then the signal is low pass filtered and sampled in an analogue to digital converter. Digitally the in-and out-of-phase quadratures are extracted.
Figure 3.13: Two separate sources are connected to the same 10 MHz reference signal. The output signals from the sources are split in two parts so that the relative phase of the two output signals can be combined in a mixer and measured by a voltmeter.

our systems the local oscillator is always set to 187.5 MHz. The mixer outputs both the upper and the lower side band, $\omega_{\text{input}} \pm \omega_{\text{LO}}$. After the mixer there is a low pass filter to extract the lower side band, $\omega_{\text{input}} - \omega_{\text{LO}}$. Then the signal is sampled in an analogue to digital converter (ADC). The digitalized signal is then digitally mixed to extract the in-and out-of-phase quadratures. From the quadratures, for instance the total output power can be calculated $P = I^2 + Q^2$.

For a system that switches between different states, such as a subharmonic oscillator or a parametric oscillator, the $I$ and $Q$ quadrature traces show this switching as jumps between signal levels. These traces are used to create a picture of the system phase space distribution. From the traces, histograms are created that clearly show the most populated states of the system. An example of quadrature raw data is shown in Fig. 4.16 and the corresponding histogram is presented in Fig. 4.14(g).

3.3.4 Phase control for double pumping

For double pumping of the doubly tunable resonator, two separate signal sources are used. The setup is described in paper B, but as a complement I also describe it here. The two microwave sources are always set to the same frequency, however their relative phase is randomly set each time the frequency is changed. Therefore, I continuously measure the relative phase of the sources to allow for post-processing of the datasets.

The output signals of the two microwave sources are split in power splitters and one part goes to the on-chip flux tuning lines and the other to a microwave mixer, see Fig. 3.13. Since the two sources always are set to the same frequency, the mixing output is a dc component, that is measured with a multimeter. This dc component is proportional to $\cos(\phi_l - \phi_r)$, where $\phi_l$ and $\phi_r$ are the phases of the two signals at the mixer inputs. This phase difference is in turn related to the actual phase difference of the flux pumps at the sample. This allows for post correction of the phase in the data.

3.4 Circuit design

There are many aspects to consider when designing a circuit, choose dimensions of the coplanar waveguide, set the strength of the coupling capacitor and dimensions of
3. EXPERIMENTAL METHODS

Figure 3.14: Simplified version of a typical layout of a SQUID-terminated \( \lambda/4 \)-resonator. To the left is the SQUID and to the right is the coupling capacitor with a metal piece behind, intended for bonding to a contact. The darker piece of metal behind the bonding pad could be left and create a huge superconducting loop. Or it could be removed and there would be no loop.

the SQUID-loop and junctions. Here I will focus on two design aspects of the circuit environment where I have gained some experience from my particular experiments.

3.4.1 Design aspects of superconducting loops

An interesting question when designing a circuit is whether to connect the ground planes behind the contact pads or not. Due to the Meissner effect, magnetic flux is expelled from the area covered by superconducting material. In Fig. 3.14, I present a typical layout of a coplanar waveguide \( \lambda/4 \)-resonator with a SQUID at the end (schematic in Fig. 2.9(b)). To the right you find the coupling capacitor and to the right of that, a piece of conductor intended for bonding, a contact pad.

To the right of the contact pad, a piece of ground plane is marked with a darker grey colour. This piece of superconductor creates a large superconducting loop. Due to the Meissner effect magnetic flux can be trapped in superconducting loops, causing noise. The larger the area of the loop, the larger the risk of trapping flux. However, a circuit embedded in a superconducting loop is also protected from external field fluctuations. An advantage with keeping the dark grey piece is that the ground plane pieces on the two sides of the resonator are better connected.

The best solution remains unknown. For other people in the group, leaving the dark piece has worked very well. However, for me there has been indications that this huge superconducting loop leads to flux noise. In Appendix B.2, two test measurements on the same design with and without a superconducting connection behind the contact pads have been tried. Both samples are measured in the same setup, with the same type of magnetic shielding. The results show that the superconducting loop indeed shields the resonator and makes it less sensitive to external magnetic fields. However, the resonance frequency measured with the shielding dark piece of ground plane is very noisy. On the other hand, when the superconducting loop is cut, the resonance frequency measured is very stable. Therefore, I avoid superconducting loops as much as I can.
Figure 3.15: Layouts of doubly tunable resonator designs. (a) The straightforward layout we get from the schematic in Fig. 2.12. (b) The design used for the main measurements where the superconducting ground plane is cut and connected with gold to ensure good electrical contact. (c) Another version of using a gold connection to break the parasitic superconducting loop is to have a piece of gold in the middle of the resonator.
3. EXPERIMENTAL METHODS

3.4.2 Design aspects of the doubly tunable resonator

A direct translation of the schematic in Fig. 2.12 gives the layout of Fig. 3.15(a). Note here how, if nothing is done, on the lower side of the resonator, there is an extra SQUID loop with a large area. The loop is formed by the resonator centre conductor and the ground plane and marked with a light green colour. Compared to the SQUID loops this parasitic loop is huge and is also a SQUID loop. To solve the problem, the ground plane is cut in two and bridged with gold, see panel (b). The purpose of the gold is to ensure good electrical contact throughout the full ground plane, however since gold remains a normal conductor also at low temperatures the remaining loop is not superconducting and does not trap flux. In panel (c) you find a second version of breaking the parasitic superconducting loop. Here the superconductivity is broken in the middle of the resonator. Since gold is a normal metal this can induce some losses, however for the first resonator mode, there is a current minima in the middle of the resonator. Minimal current means that the losses should be small.

Even though the loop is not superconducting there remains a loop through the resonator and the lower part of the ground plane. This loop is big and could potentially inductively couple to other structures on-chip or in the sample box.
4

Results

Here I present measurement results from several different samples. The key findings are presented in the papers which are attached at the end of this thesis. However, in the thesis I extend the picture by presenting results from more samples and bias points. There will be some analysis and discussion on different implications. I start with characterization of resonators and continue with fast modulation of the SQUID nonlinearity. The characterization is done on samples without SQUIDs as well as single-SQUID and double-SQUID tunable resonators. For the doubly tunable resonator, I discuss different designs, loss mechanisms and dc crosstalk. In the fast modulation section, I present measurement results that are extensions of the Paper A and B contents on different types of period multiplication subharmonic oscillations. In section 4.3.5, additional information on the doubly tunable resonator that supports the claims in Paper C are presented. In section 4.4 I do a short review of the results of Paper E and F on microwave manipulation in a tunable coupling system.

4.1 Resonator characterization

Characterisation measurements are done using a vector network analyser (VNA) in reflection setups, see Fig. 3.8. The measured data consist of both magnitude and phase of the reflection coefficient as a function of frequency. Fitting the resonance with Eq. (2.20) allows for extraction of Q-values and resonance frequency.

4.1.1 Linear resonator

The linear resonator is a plain transmission line resonator without any SQUID, i.e. without any nonlinearity. The sample I show data from is a \(\lambda/2\)-resonator, with the design presented in Fig. 3.15(a), but with continuous metal instead of SQUIDs. It is a niobium resonator on a silicon substrate, fabricated using the recipe in appendix A.4. The measurements were performed in our \(^3\)He cryostat at roughly 300 mK.

The raw data of a reflection measurement is presented in Fig. 4.1 as blue dots. The
4. RESULTS

![Figure 4.1](image1.png)

Figure 4.1: An example of a resonance data trace (blue dots) with corresponding fit (red line). This specific one is for a probe power of $-117$ dBm, the extracted resonant frequency and $Q$-values are $ω_r/(2\pi) = 4.9796 \text{ GHz}$, $Q_{ext} = 9.4 \cdot 10^4$ and $Q_{tot} = 8.2 \cdot 10^4$. In (a) and (b) you find the magnitude and the phase of the reflection coefficient. (c) The reflection coefficient plotted in the polar plane.

![Figure 4.2](image2.png)

Figure 4.2: Probe power dependence of the quality factors. (a) External quality factor. (b) Total quality factor. (c) Internal quality factor. The increase with larger probe power indicates that one of the dominating loss factors could be two level fluctuators.

different panels show (a) the magnitude response, (b) the phase response and (c) a polar plot of the real and imaginary parts of the reflection coefficient. Here some processing of the data has been done, the background has been normalised to ensure that $|S_{11}| = 1$ far from resonance and the electrical delay in the measurement setup has been compensated. The extracted resonance frequency and $Q$-values at low power are $ω_r/(2\pi) = 4.9796 \text{ GHz}$, $Q_{ext} = 9.4 \cdot 10^4$ and $Q_{tot} = 8.2 \cdot 10^4$. The internal $Q$-value is calculated using Eq. (2.12) to $Q_{int} = 6.3 \cdot 10^5$.

The data in Fig. 4.1 represent a probe power of $-117$ dBm. In Fig. 4.2, extracted $Q$-values are presented for many different probe power values. The power scale spans from single photon level up to millions of photons. As seen, the total $Q$-value increases with the probe power while the external $Q$-values remains the same, which means that the internal $Q$-value also is increasing with probe power. This is a signature that the main loss mechanism is the two-level fluctuators in the substrate [42, 77]. At larger probe power, the two-level fluctuators are saturated and interact less with the resonator and
4.1. Resonator Characterization

-150 dBm  -145 dBm  -140 dBm  -135 dBm  -130 dBm  -125 dBm  0

<table>
<thead>
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5.490 5.495
0
-2 
2

Figure 4.3: Reflection measurement on a SQUID-terminated λ/4-resonator biased close to zero flux. (a) Magnitude response versus probe power. (b) Linecuts from the magnitude response. (c) Linecuts from the corresponding phase response.

cause less less loss.

4.1.2 Resonator with a nonlinear element

In a reflection measurement, the resonator with a nonlinear element behaves just as a plain linear resonator, for low probe power. However, when the probe power is increased, the nonlinearity affects the response. The nonlinear elements treated in this thesis are, either a single Josephson junction or two Josephson junctions in parallel, a SQUID. By low probe power I assume a probe power that generates a current well below the critical current of the Josephson junction(s).

As an example of nonlinear effects, I show data from a λ/4-resonator terminated with a SQUID, Fig. 2.9(b), made of aluminium on a silicon substrate, sample #3 in table 4.1. For zero flux bias, the magnitude response is plotted versus probe power in Fig. 4.3(a). The reflection measurement is performed using a VNA. Linecuts of the magnitude response are plotted in (b), together with the corresponding phase traces in (c). For low probe power, ≲ −140 dBm, the magnitude response is barely noticeable. Together with the fact that the phase does a 2π-wrap this indicates an overcoupled resonator. Ideally, the magnitude response would be a dip or just flat, however there is a S-shape. This S-shape is probably due to an impedance mismatch between the transmission line and the resonator. Above −140 dBm the depth of the magnitude response starts to grow. At larger probe power the dip appears even deeper and also shifted towards red (negative) detuning. Also, the phase response shows an asymmetric behaviour for the larger probe power values. This asymmetry is due to the bifurcation described in section 2.3.2. When the probe power is increased, the photon population in the resonator increases, and the generated current in
4. RESULTS

4.1.3 Measurements of modes outside the available measurement band

To explore the resonator spectrum outside the setup measurement band, parametric up-conversion can be used, see section 2.4.5. A weak probe signal is applied to one of the measurable resonator modes, at the same time as we fluxpump the SQUID at the difference frequency between the measurement mode and another resonator mode, photon conversion between the modes is detected as an avoided level crossing [68]. The probe signal is applied to the input port of the reflection setup, while the flux pump is applied through an on-chip flux pump line. The probe mode frequency is known and then the difference frequency between the modes can be extracted. Hence, the wanted mode frequency is simply $\omega_p + \omega_1 = \omega_2$. A measurement example is shown in Fig. 4.4. The data is taken from a SQUID-terminated $\lambda/4$-resonator, Fig. 2.9(b). It is the same example as presented in Paper A.
4.2 Static biasing of a tunable resonator

By sweeping the applied flux bias of the samples, the resonance frequency dependence on magnetic flux can be explored. Depending on the resonator design, different behaviour is expected.

Firstly, as a reference, Fig. 4.5 shows the frequency tuning of a plain niobium $\lambda/4$-resonator without any SQUID, solely a transmission line of niobium. To first order, a plain resonator should not show any flux dependence. Nevertheless, there is a small shift in resonance frequency in Fig 4.5, we attribute this to a modification of the kinetic inductance of the resonator. It should however be noted, that the measured resonance shift is of the order of kHz while the resonance frequency is around 5 GHz. The magnetic field strength is calibrated by a SQUID-tunable resonator placed on the same chip. The measurement was performed in a dilution cryostat at 10 mK.

Another resonator type, that should not be sensitive to magnetic fields, is a $\lambda/4$-resonator terminated by a single Josephson junction. The sample under study is a niobium resonator with aluminium SQUIDs and measurements were performed at 10 mK. Interestingly, the measurement results show that the resonance frequency is sensitive to magnetic flux, see Fig. 4.6. Again, the magnetic field strength is calibrated using a SQUID-tunable resonator placed on the same chip. Compared to the plain resonator, Fig. 4.5, the effect is large however still small compared to the resonance frequency that is around 5 GHz. This magnetic field sensitivity could possibly be explained by flux penetrating into the junctions, reducing the Josephson inductance. The larger jumps at 0.22 and 0.3 mT also indicate that there was some rearrangement of trapped flux close to the junction.

4.2.1 Quarter wave-length SQUID-terminated resonators

Here I will show measurement data from some different $\lambda/4$-resonators, see schematic in Fig. 2.9(b). Most of them have nominally symmetric SQUIDs while one sample has an asymmetric SQUID. The spectra for these resonators are given by the dispersion relation in Eq. (2.28), however for the case of an asymmetric SQUID the inductive term has to
be rewritten accordingly. The first resonator mode is always probed directly through a reflection measurement. For one sample, the second mode is also directly probed while the other samples required second mode extraction through frequency up-conversion, as described in section 2.4.5.

**Symmetric SQUID**

For a symmetric SQUID, the inductance is given in Eq. (2.9), this is the case assumed in the dispersion relation, Eq. (2.28). In Fig. 4.7, the data and corresponding fits of three different samples are shown. Extracted parameters of the samples are given in Table 4.1. Sample #1 is the same as presented in Paper A and sample #2 is very similar to #1, only minor differences in resonator length and junction design, which results in different participation ratios. Both these samples are fabricated using the recipe in section A.1 and have their lowest mode placed around 5 GHz. This means that the second mode is around 15 GHz and thereby outside the available measurement band (maximum 3 – 12 GHz). Hence, the data points for the second mode on these samples are measured by parametric frequency up-conversion. Sample #3, on the other hand, is designed to allow for direct measurements of both the first and the second mode. This sample is fabricated using recipe A.2. Then there is a fourth sample. This sample is fabricated with the same recipe as #1 and #2 and the design is very similar apart from the fact that resonator #4 is longer to have a lower resonance frequency. In this sample the second mode is below 12 GHz and can be directly probed in our measurement setup. The data has not been fitted, hence the lack of extracted parameters in the table.

The nonlinearity of the SQUID gives rise to a non-equidistant resonator spectrum. The modes have different separation. For the fits in Fig. 4.7 the anharmonicity, $3\omega_1 - \omega_2$ is plotted in Fig. 4.8. Here, both sample #1 and #2 behave very similar, starting at some positive value at zero flux, and then cross zero at larger flux bias. In contrast, sample #3 starts at negative anharmonicity at zero flux, and stays below zero for the full flux period. The reason for the difference in anharmonicity sign is found in the junction design. Compared to sample #1 and #2, the junctions in sample #3 have very small metal layer

![Figure 4.6: Magnetic flux sensitivity of a resonator terminated by a single Josephson junction.](image)

4. RESULTS
Figure 4.7: The two first modes of three different samples. The dots correspond to data points and the lines are the best found fits. Specifications of the different samples are found in Table 4.1.
4. RESULTS

Table 4.1: Specification of the $\lambda/4$ SQUID-tunable resonators presented in this thesis. Sample #1 – 4 have symmetric SQUIDs while sample #5 have an asymmetric SQUID. The properties stated in the table are, appendix reference to the recipe used for fabrication, the resonator length, $d$, the zero flux resonance frequency $\omega_1(0)$, the SQUID critical current, $I_c$ (for sample #5 that has an asymmetric SQUID the critical current of the respective junctions are given), the SQUID capacitance, $C_{SQ}$, the inductive participation ratio, $\gamma_0$, the inductance and capacitance per unit length, $L_0$ and $C_0$, the spectrum anharmonicity at zero flux, $3\omega_1(0) - \omega_2(0)$, the linewidth, $2\Gamma_1$ and the Q-values, $Q_{ext,1}$ and $Q_{tot,1}$.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Fabrication recipe</th>
<th>$d$ [mm]</th>
<th>$\omega_1(0)/2\pi$ [GHz]</th>
<th>$I_c$ [$\mu$A]</th>
<th>$C_{SQ}$ [fF]</th>
<th>$\gamma_0$ [%]</th>
<th>$L_0$ [$\mu$H/m]</th>
<th>$C_0$ [nF/m]</th>
<th>$(3\omega_1(0) - \omega_2(0))/2\pi$ [MHz]</th>
<th>$2\Gamma_1(0)$ [MHz]</th>
<th>$Q_{ext,1}(0)$ [$10^3$]</th>
<th>$Q_{tot,1}(0)$ [$10^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>A.1</td>
<td>5.1</td>
<td>5.504</td>
<td>1.90</td>
<td>86</td>
<td>7.7</td>
<td>0.44</td>
<td>0.16</td>
<td>136</td>
<td>0.38</td>
<td>19</td>
<td>14.5</td>
</tr>
<tr>
<td>#2</td>
<td>A.1</td>
<td>5.2</td>
<td>5.225</td>
<td>1.47</td>
<td>63</td>
<td>10.7</td>
<td>0.41</td>
<td>0.17</td>
<td>86</td>
<td>0.49</td>
<td>11.4</td>
<td>10.7</td>
</tr>
<tr>
<td>#3</td>
<td>A.2</td>
<td>7.9</td>
<td>3.557</td>
<td>0.85</td>
<td>~ 25</td>
<td>10.6</td>
<td>0.45</td>
<td>0.14</td>
<td>-56</td>
<td>0.12</td>
<td>33.8</td>
<td>29.4</td>
</tr>
<tr>
<td>#4</td>
<td>A.1</td>
<td>6.8</td>
<td>4.035</td>
<td>1.05/0.31</td>
<td>11.5</td>
<td>10.6</td>
<td>0.41</td>
<td>0.17</td>
<td>21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td>A.1</td>
<td>5.1</td>
<td>5.244</td>
<td>1.05/0.31</td>
<td>230</td>
<td>11.5</td>
<td>0.41</td>
<td>0.17</td>
<td>21</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

overlaps and thinner oxide layer. This results in similar values for SQUID inductance, however the capacitance becomes small, due to the small overlaps. The capacitance is very small also compared to the geometric capacitance of the resonator, which is larger for sample #3 simply because the resonator is longer. Therefore, it is difficult to extract a number, a simple comparison of the junction area gives that sample #3 have roughly 40% of the junction area of sample #2. That correspond to roughly a capacitance of 25 fF for sample #3.

Asymmetric SQUID

For an asymmetric SQUID, the inductance is given by the more complicated expression, Eq. (2.8). Hence, the inductive term in the dispersion relation, Eq. (2.28) needs to be replaced. The dispersion relation for the asymmetric SQUID is written as

$$k_n d \tan k_n d = \frac{1}{\gamma_0} \sqrt{\cos^2 \Phi \pi \Phi_0 + \left( \frac{\Delta I_c}{I_c} \right)^2 \sin^2 \frac{\Phi \pi}{\Phi_0} - c(k_n d)^2}. \quad (4.1)$$

In Fig. 4.9, I present resonance data extracted from sample #5. The sample is very similar to the symmetric SQUID samples #1 and #2. The difference is that this resonator has an asymmetric SQUID, $I_{c,1} \neq I_{c,2}$. Furthermore, the asymmetric SQUID gives a minimum on the tuning curve set by the difference between the critical currents of the two junctions, while the maximum on the tuning curve is set by the sum of the junction critical currents. The red line corresponds to the best fit found using Eq. (4.1).
4.2. STATIC BIASING OF A TUNABLE RESONATOR

Figure 4.8: Spectrum anharmonicity, extracted from the fits in Fig. 4.7 for the three samples. Note how sample #3, in contrast with the other samples, has a negative anharmonicity for all flux values.

Figure 4.9: Static tuning of the resonance frequency for sample #5 which has an asymmetric SQUID. Note how the resonance frequency has a minimum (set by the difference between the critical currents of the two junctions). The red line is calculated using the parameters $I_{c,1} = 1.05 \mu A$, $I_{c,2} = 0.31 \mu A$, $C_{SQ} = 230 fF$, $L_0 = 0.41 \mu H/m$, $C_0 = 0.17 nF/m$. 

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4.2.2 The doubly tunable resonator

In the double SQUID resonator we would like the two SQUIDs to be controlled independently, to have the tuning pattern as in Fig. 2.13. Therefore we use on-chip flux tuning lines, one for each SQUID. A brief overview and discussion on flux tuning lines can be found in section 3.3.1.

Preferably an on-chip tuning line should give a strong mutual inductance between the SQUID and itself but a minimal coupling directly to the resonator. In addition, for the double SQUID resonator, the mutual inductance should be strong between a SQUID and its own dedicated flux tuning line, but weak between a flux tuning line and the opposite SQUID. Aiming for larger quantum networks, crosstalk is a highly important issue. Here follow some of my experiences, working with the doubly tunable resonator.

For the doubly tunable resonator (DTR) there are several aspects to consider in designing the circuit and the setup, one is discussed in section 3.4.2. Many test measurements of different designs have been performed and here I present some of them. They are represented by the micrographs in Fig. 4.10. These five samples are:

- **DTR #1**: Flux tuning line design (a). No cut in the ground plane results in a huge parasitic SQUID loop, see Fig. 3.15(a). The sample is made of aluminium on silicon substrate, recipe A.3.

- **DTR #2**: Flux tuning line design (b). The ground plane is cut by the flux tuning line, hence there is no parasitic SQUID loop. The flux tuning lines are grounded by...
bonding from the chip contact pads to the PCB ground plane. The sample is made of aluminium on silicon substrate, recipe A.3.

- DTR #3: Flux tuning line design (a). No parasitic SQUID loop due to gold-filled slot in the resonator middle, see Fig. 3.15(c). The sample is made of niobium with aluminium SQUIDs on a silicon substrate, recipe A.4.

- DTR #4: Flux tuning line design (c). The ground plane has a gold filled slot in the middle, opposite to the coupling capacitor to avoid a parasitic SQUID loop, see Fig. 3.15(b). The flux tuning line is grounded on the PCB ground plane. The sample is made of niobium with aluminium SQUIDs on a sapphire substrate, recipe A.1.

- DTR #5: Flux tuning line design (c), the ground plane has a gold filled slot in the middle, opposite to the coupling capacitor to avoid a parasitic SQUID loop, see Fig. 3.15(b). The flux tuning line continues off-chip and the line is grounded at room temperature. The sample is made of niobium with aluminium SQUIDs on a sapphire substrate, recipe A.1.

The idea with flux tuning line design (a) is that the two perpendicular arms in the flux tuning line end will have opposite inductive coupling to the resonator, so that the parasitic inductive coupling directly between the flux tuning line and resonator is minimal. In design (b) the direct coupling between the flux tuning line and the resonator is large, test measurements show that this coupling severely limits the achievable Q-values. To minimize the parasitic direct coupling, but retain the possibility to lock the current in a closed circuit design, (c) was created. Simply a transmission line perpendicular to the resonator.

Measurement results for the different samples can be found in Fig. 4.11 and 4.12. In the plots of the first figure, horizontal and vertical axes correspond to the current applied to the left and right flux tuning line respectively. The current applied to the flux tuning line scales to applied magnetic flux via the mutual inductance. The resulting flux in the SQUID loop can be written \( \Phi_{dc,l/r} = M_{r/l}I_{l/r} + M_{ct,l/r}I_{l/r} + \Phi_{offset,l/r} \), where the mutual inductance between a SQUID and its own flux tuning line is denoted \( M_{l/r} \) and the mutual inductance between a SQUID and the remote flux tuning line is \( M_{ct,l/r} \). The current applied to the separate flux tuning lines are denoted \( I_{l/r} \) and a possible flux offset given from the cooldown conditions is given by \( \Phi_{offset,l/r} \).

In Fig. 4.11(a), I present the measurement results from sample DTR #1, the resonance frequency tunes in an elliptic pattern. This is an indication that the crosstalk is large. This could for instance be the current travelling in the ground plane from one side of the resonator to the other one. Though, note how more flux tuning periods are visible in the bottom left and top right corner. Note also that in this sample the parasitic SQUID loop is not broken, which means that the tuning could be different from expected.

A design that both avoids uncontrolled currents in the ground plane and cuts the parasitic superconducting loop is the flux tuning line design in Fig. 4.10(b). As the test measurement on DTR #2 presented in Fig. 4.11(b) shows, this results in very small crosstalk. Furthermore, the same result can be achieved when the flux tuning lines are grounded on chip above the SQUID, see Fig. 4.10(d). Note that in the design tried, the grounding of the flux tuning lines is in a direction where the uncontrolled ground currents
Figure 4.11: Extracted first mode resonance frequencies, for static biasing of both the left (horizontal axis) and right (vertical axis) SQUID. (a) Sample DTR #1. The measurement show an elliptic shape, which can be interpreted as a signature of the parasitic SQUID loop and/or large crosstalk. (b) Sample DTR #2. The pattern is fairly quadratic which means that the crosstalk is small and there is good individual control of the two SQUIDs. (c) Sample DTR #3. The elliptic shape indicates large crosstalk. The reason for this large crosstalk is unknown. Since there is no parasitic SQUID loop, that should not be the problem. (d) Sample DTR #4. Note how the scale on the current is suddenly micro Ampere instead of milli Ampere. In addition there is no sign of a second flux period. This is an indication that the tuning is rather a saturation of the SQUID critical current than flux tuning.
4.2. STATIC BIASING OF A TUNABLE RESONATOR

Figure 4.12: Static biasing of a doubly tunable resonator using the flux tuning line design in Fig. 4.10(c). This data is from Paper C and the sample DTR #5. The flux tuning lines continue off-chip and up to room temperature where they are grounded. (a) Static biasing of the first resonator mode. (b) Example line cut from (a) showing the good fit to Eq. (2.33).

have long paths to reach the remote SQUID. From a crosstalk point of view, this design is a winner. However, from a loss point of view it is bad, test measurements typically show very low internal Q-value due to the direct coupling between the flux tuning line and the resonator.

The sample DTR #3 is special because it has a gold slot in the resonator middle (Fig. 3.15). This cuts the parasitic SQUID loop, but as seen in Fig. 4.11(c), not the crosstalk. Note that there is a second period visible which indicates that the frequency tuning mechanism is indeed flux tuning. The crosstalk could be because of uncontrolled currents in the ground plane that travels from the grounding of one flux tuning line to the other.

To solve the crosstalk problem and also have low loss, the on-chip flux tuning line can be placed perpendicular to the resonator, Fig. 4.10(c). For sample DTR #4, the flux tuning lines continue off-chip and are grounded on the sample box PCB. The measurement result is found in Fig. 4.11(d). In this measurement the SQUIDs can not be controlled individually. Also no second period is visible. Note also that the currents applied on the flux tuning lines are orders of magnitude lower than in the other panels. Micro ampere is of the order of the SQUID critical current. This indicates that the frequency tuning is rather something that saturates the SQUID than tunes it. Likely, the dc current reach the resonator and actually saturates the SQUIDs.

However, if the perpendicular flux tuning line design in Fig. 4.10(c) is used, but the circuit is set up so that the dc current is transported back up to room temperature and grounded there, the SQUIDs can be biased individually, see Fig. 4.12. This is the same sample and figure that are presented in Paper C. In the thesis this sample is known as DTR #5.

One conclusion from all my double SQUID dc-tuning tests is that, to be safe, you
want to transport the current away from the sample and preferably even back to room
temperature. Another reflection is that the design in Fig. 4.10(b) is very convenient to
use since it removes the need of gold slots to sort out the parasitic SQUID loop. However,
as stated above, the coupling directly between the transmission lines causes large losses.
A possible solution would be to minimize the part where the flux tuning line and the
resonator are parallel, which is not fully done in Fig. 4.10(d).

4.3 Nonadiabatic modulation of a SQUID

As discussed in the theory section there are two methods to modulate the SQUID
nonlinearity, either by current driving or by flux pumping. Here I present a larger library
of measurement results that did not fit into Papers A, B and C. I start with subharmonic
oscillations and continue with a standard parametric oscillator and the doubly tunable
resonator.

The power scales in this section correspond to signal generator power subtracted
by the installed attenuation. This means that the power scales in principle should be
comparable, though it should be remembered that different measurements have been done
using different setups and the cable losses may vary.

In order to have subharmonic oscillation excitations rather than SQUID excitations,
it is important that the plasma frequency of the SQUID is substantially larger than
the drive/pump frequency $\omega_{p,SQ} = \frac{1}{\sqrt{L_{SQ}C_{SQ}}} \gg \omega_{d/p}$. Given the numbers from the
fits in the previous section, the SQUID capacitance is less than 100 fF and the SQUID
inductance around zero flux is of the order of 0.3 nH. Then the SQUID plasma frequency
is around 30 GHz. Hence, for third order subharmonic oscillations, the plasma frequency
should be at least a factor two larger than the drive frequency at zero flux, assuming a
fundamental resonator mode around 5 GHz. For higher order multiples though the drive
will be closer to the plasma frequency and possibly interfere. It should also be noted that
the plasma frequency decreases with flux which could make the problem larger at larger
flux values.

4.3.1 Period tripling subharmonic oscillations

I discovered the subharmonic oscillations while working with a flux pumped doubly
tunable resonator, however to reduce the number of unknown parameters, it has been
investigated mostly in a current driven $\lambda/4$ SQUID-terminated resonator. In this section
I focus on current driving, an external drive signal is applied to the resonator input
port, Fig. 2.18. Since theory predicts existence of the oscillations at red detuning, the
measurement frequency, $\omega$, is set to sweep from large negative detuning up to around
zero. The drive frequency is fixed to 3$\omega$. Starting from the measurement frequency is
important here. The measurement frequency can not be set to 1/3 of the drive since then,
for all frequencies not directly dividable by three, there will be rounding errors. Rounding
errors cause improper phase matching and effectively a phase drift in the measurement.
A figure illustrating this is found in appendix D.1.

The current driven subharmonic oscillations can be detected for many different flux
bias values. In Fig. 4.13 I present regions, in which the subharmonic oscillations can be
4.3. NONADIABATIC MODULATION OF A SQUID

Figure 4.13: Current driven subharmonic oscillations measured in sample #1, see table 4.1. This is also the same sample as presented in Paper A. The panels correspond to (a) $0 \Phi_0$, $\omega_1/2\pi = 5.504 \text{ GHz}$ (b) $0.1 \Phi_0$, $\omega_1/2\pi = 5.483 \text{ GHz}$ (c) $0.2 \Phi_0$, $\omega_1/2\pi = 5.409 \text{ GHz}$ (d) $0.3 \Phi_0$, $\omega_1/2\pi = 5.234 \text{ GHz}$ (e) $0.4 \Phi_0$, $\omega_1/2\pi = 4.747 \text{ GHz}$. Note that the detuning range is much larger here than in Fig. 4 of Paper A. The measurement temperature was 12 mK.
found, for sample #1 flux biased at 0, 0.1, 0.2, 0.3, and 0.4 Φ₀. For the four first bias points, the regions are fairly similar, they start at some finite red detuning value and persists in a narrow power interval towards larger red detuning. The wavy structure of the regions we attribute to background structures in the measurement setup. The oscillation intensity increases with increased red detuning. However, the larger the flux, the lower the intensity of the oscillations, see the difference in the colour scales. When the flux is increased to 0.4 Φ₀ the subharmonic oscillations are suddenly very weak in intensity and they disappear already at small detuning. Two reasons for this behaviour at large flux are easy to find. First, at this large flux the SQUID critical current is suppressed and the threshold for nonlinear effects is low. Secondly, the anharmonicity of the resonator spectrum is here very small, which decreases the predicted size of the region of existence for the subharmonic oscillations.

In Paper A we present a selection of zero-flux-histograms for sample #1. Here I will show more. For sample #1, more histograms measured at zero flux are found in Fig. 4.14, and they are extracted at the measurement points indicated by the white circles in Fig. 4.15.

In Fig. 4.14(a) the ground state of the system is presented. This is the state, in which I always find the system above and below the subharmonic region. The size of the blob gives a measure of the system noise level. At the lower boundary of the subharmonic oscillation region there is a competition between the ground state in the middle, and the triplet excited state. This can be seen in Fig. 4.14(b,d,f). Also in (j) there is a faint spot in the middle. In the middle of the subharmonic oscillation region, the excited state wins and the histograms show only the triplet state, see Fig. 4.14(c,e,g,h,k). Interestingly, when the detuning is increased, the system starts showing a competition also at the upper region boundary, see Fig. 4.14(l). Also possibly Fig. 4.14(i) could be a mixture of the ground and the excited state, however the limited resolution of the measurement and the low signal level makes it impossible to distinguish whether there are three or four states involved.

The raw data for the histogram in Fig. 4.14(g), the in- and out-of-phase quadratures are presented in Fig. 4.16. However, only for the first 3000 samples. The sampling rate is 100 kHz. Note how the traces switch between three different levels. From these traces, the system switching rate can be extracted by counting the number of switches. For this specific measurement point, the extracted switching rate is around 1 kHz, however the switching rate depends strongly on the exact measurement parameters.

In Fig. 4.17, histograms for sample #1 with flux bias 0.18 Φ₀ are presented. The measurement points are indicated in Fig. 4.18. A comparison of these histograms and the ones of Fig. 4.14 show that the behaviour is very similar for 0.18 Φ₀ compared to zero flux. At the lower boundary, see Fig. 4.17(b,d,g,j), the histograms feature the triplet excited state together with the ground state. However, in (d) and (j) the ground state is very weak. Inside the region of subharmonic oscillations, and at the upper boundary, only the triplet state is visible, see Fig. 4.17(c,e,h,i,k,l). In (i) the measurement point is very close to the upper edge of the subharmonic region, therefore the oscillator intensity is very low and the histograms becomes a triangle. In this figure, I also show a background histogram from above the upper boundary, panel (f). It is found to be very similar to the background level below the subharmonic region, panel (a).
Figure 4.14: Histograms measured at zero flux bias at the points indicated by the white circles in Fig. 4.15. The sampling rate is 100 kHz and the number of samples $5 \cdot 10^5$. 
4. RESULTS

Figure 4.15: Subharmonic oscillation region for sample #1 at zero flux bias and with a resonance frequency $\omega_1/2\pi = 5.504$ GHz. The white circles indicate the positions of the histograms in Fig. 4.14.

Figure 4.16: Raw data for the quadrature traces forming the histogram in Fig. 4.14(g). The sampling rate is 100 kHz and here the first 3000 out of $5 \cdot 10^5$ samples are displayed.
Figure 4.17: Histograms measured for sample #1 at 0.18 \( \Phi_0 \) at the points indicated in Fig. 4.18. The traces are sampled at a frequency of \( f_s = 100 \) kHz and consist of \( 5 \cdot 10^5 \) samples.
4. RESULTS

The subharmonic oscillations have been observed in several different samples. In Fig. 4.19, histograms from sample #2 flux biased at 0.17 Φ₀ are presented. The measurement points are indicated in Fig. 4.20. In this data set, all the histograms are taken at the same frequency but at different drive power. Hence, here the evolution of the system from the single ground state in (a) to the excited state can be followed. In (b) and (c) we see, in the competition between the ground state and the excited state, how the excited state first is weak and then grows stronger as the drive power is increased. Then, when we reach the middle (d) and the upper part (e-f) of the oscillation region the quadrature histograms show only the excited state. When the drive power is increased the amplitude of the state decreases.

In sample #1 and #2 the anharmonicity of the spectrum is positive, i.e. $3\omega_1 - \omega_2 > 0$, and for not too big flux bias values, the drive frequency is placed above the second mode, see Fig. 4.8. Contrarily, sample #3 has a negative anharmonicity for all flux bias values. The theory (Fig. 2.21 and 2.22) still predicts existence of the triplet excited state though in a smaller region, and indeed it can be measured, see Fig. 4.21. Again, the combination of the ground state and the triplet state in panel (b) is measured at the low drive power boundary, and the pure triplet state in panel (c) is measured in the middle of the subharmonic oscillation region.

In Paper A we show that we have a good agreement between theory and experiment for the case of positive anharmonicity. However, the negative anharmonicity regime is not explored. It can be noted that the subharmonic oscillation regions for sample #1 and #2 spread over larger ranges of both drive power and detuning compared to sample #3. This could be because of the spectrum anharmonicity. In the theory plots in Fig. 2.21 and 2.22, the regions of predicted stability are smaller for the negative anharmonicity than for positive. Another difference between the samples is that sample #3 is fabricated with a different technique and is of better quality. The better Q-value leads to narrower
4.3. NONADIABATIC MODULATION OF A SQUID

Figure 4.19: Histograms from sample #2 biased at 0.17 \( \Phi_0 \) and the resonance frequency \( \omega_1/2\pi = 5.149 \text{ GHz} \). The measurement points are indicated in Fig. 4.20. The data is sampled at 100 kHz and consist of \( 4 \cdot 10^5 \) samples.

Figure 4.20: The white circles indicate the measurement points for the histograms in Fig. 4.19. The data is measured on sample #2 biased at 0.17 \( \Phi_0 \).
4. RESULTS

modes, possibly this could also contribute to the more narrow subharmonic oscillation effect. Lastly, there is one difference between the different measurement setups, in which the samples have been measured, the sample boxes are different. Both sample #1 and #2 are measured using old style sample boxes with PCBs, these boxes are known to host parasitic modes. These background modes could facilitate the down-conversion process, similar to the increased efficiency induced by the second resonator mode. In contrast, sample #3 is measured in a new sample box, known to be free of spurious modes up to 12 GHz, which is higher than $3\omega_1$. Hence, the reduced size of the subharmonic region could therefore possibly also be related to the absence of parasitic box modes.

Also sample #1 and #2 has regimes with negative anharmonicity, however no subharmonic oscillator effect has been observed in these cases. On the other hand, the asymmetric SQUID sample #5, characterized in Fig. 4.9, has shown weak subharmonic oscillations at negative anharmonicity. This sample has positive anharmonicity at zero flux but then it decreases and is negative around half a flux quantum. This is where I actually have measured a weak subharmonic response, at around half a flux quantum.

Analysis of a subharmonic region

In Paper A we show how we can fit three line cuts of Fig. 4.15 which is measured at zero flux bias. In Fig. 4.22, I show a fit of the subharmonic region in Fig. 4.18, measured at $0.18\Phi_0$. The first mode resonance frequency is $\omega_1/2\pi = 5.432$ GHz and the Q-values are very similar to the zero flux values given in Paper A.

The drive power applied from a microwave source is related to the external drive field
Figure 4.22: Output power of the subharmonic oscillator versus drive power for sample #1 at the bias point $0.18 \Phi_0$. The dotted traces correspond to three line cuts of Fig. 4.18, $\delta_1 = -3.5$, $-7$, $-10.5$ MHz. The green traces correspond to a fit.

$$|B_2|^2$$ as

$$P_d = 3\hbar\omega|B_2|^2 10^{Att/10},$$

where $Att$ denotes the total attenuation of the microwave signal from the source to the sample. Furthermore, the external drive populates the second mode following Eq. (2.59), which gives

$$P_d = 3\hbar\omega (3\omega_1 - \omega_2)^2 r_2^2 10^{Att/10},$$

where $\Gamma_{ext,2} = \omega_2/2Q_{ext,2}$. The second mode population acts as a parametric pump on the first mode which gives rise to a population in the first mode according to Eq. (2.56).

The generated signal in the first mode is then measured as

$$P_{out} = \hbar\omega r_1^2 2\Gamma_{ext,1} 10^{G/10},$$

where $G = 66$ dB denotes the total gain on the output line. Combining these three expressions above gives one single fitting parameter, $X = Q_{ext,2} 10^{Att/10}$. Other resonator specific parameters are found in Table I in Paper A. The fit in Fig. 4.22 is for $X = 1.5\cdot10^{11}$. A discrepancy between the theory curve and the measurement data can be seen in Fig. 4.22, at the low power side of the subharmonic region. As discussed in Paper A, this is because the theory only predicts the existence of the excited state and the ground state but not which one the system prefers.

The six-blob histogram

As a peculiar detail I found a very special form of the third order subharmonic oscillations in sample #2, see Fig. 4.23 and Paper B. For small detuning values, the system shows either a combination of the triplet excited and ground state, or only the excited state. However, for larger values of the detuning, the middle state suddenly divides into three states. We have not come up with an explanation for this effect, but as seen here, it
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(a) Subharmonic oscillation region, the white circles indicate the measurement points of the histograms in panel (b-d). Note the special feature in (b), where the state is divided up into two distinct triplets.

appears at larger detuning where the signal levels are stronger. The effect could be due to some higher order nonlinearity triggered by the strong subharmonic oscillator intensity.

Subharmonic oscillations with a probe signal

The configuration is described in Fig. 4.24. There is an external current drive signal at $\omega_{\text{drive}} = 3\omega$ and the measurement is set to be $\omega_{\text{meas}} = \omega$. In addition, a probe signal is applied at $\omega_{\text{input}} = \omega + \delta$. The probe and the drive signals are applied through the same cable, they are combined in a microwave combiner at room temperature.

Figure 4.25 shows measurements with zero detuning between the measurement frequency and the probe signal. The three different panels represent increasing probe power. Note how all the three states have similar intensity in panel (a). In (b), the top right state has decreased in intensity and in (c) one state is gone, and of the remaining two states, one is bright and one is weak. The system, without a probe signal, is defined by the phase portrait in Fig. 2.20, which illustrates a metapotential with three excited states and one middle ground state. The effect of the probe signal in Fig. 4.25 indicates that the metapotential is tilted, the stronger the probe, the larger the tilt. In (a), when the probe power is weak, the metapotential remains flat and all three states are populated. When the probe power is increased, the metapotential gains a small tilt and as seen in
4.3. NONADIABATIC MODULATION OF A SQUID

4.3.1 Measurement of subharmonic oscillations

Figure 4.24: Measurement configuration for subharmonic oscillations measured with an applied probe signal. The measurement is fixed at frequency \( \omega \) and the drive at \( 3\omega \). An additional probe signal is applied at \( \omega + \delta \).

Figure 4.25: Quadrature histograms of a subharmonic oscillator with an additional probe signal applied. The detuning between the probe signal and the measurement frequency is is set to zero. The three different panels are taken for three different powers of the probe signal, (a) \(-146\) dBm, (b) \(-136\) dBm and (c) \(-126\) dBm. They are all sampled at 100 kHz.

(b) one state becomes less populated, an indication that it is energetically less favourable. Furthermore, in (c) when the probe power is even higher, the tilting effect grows stronger. One state remains strong, one is weak and one barely visible.

More interesting effects can be created, when the detuning between the measurement signal and the probe is set to a finite value. In Fig. 4.26, \( \delta = 1\) Hz and this gradually deforms the states into crescent like features.

4.3.2 Current driving vs flux pumping

In Fig. 4.27, I present a comparison between (a) driving the SQUID with an external signal applied to the input port, and (b) by pumping through the on-chip flux pump line, for modulation at \( 3\omega \). The two graphs show similar features. The onset of oscillations occur at in principle the same frequency. The increase in oscillation intensity with detuning is very similar. The points at which the response disappears and reappears occur at the same frequencies in both cases. One difference is the wiggles of the oscillation region. Likely, these wiggles are due to background resonances in the setup and in the sample box. Since the current drive and the flux pump signal goes through different cables, they are likely to show different backgrounds. This flux pumping result can be explained either by an asymmetry in the SQUID (as discussed in Paper B) or by a parasitic coupling between the on-chip flux pump line and the resonator. Even though the SQUID is designed to be symmetric, due to imperfections in the fabrication process, a small asymmetry is not
4. RESULTS

Figure 4.26: Quadrature histograms of a subharmonic oscillator with an additional probe signal applied slightly detuned from the measurement frequency. The detuning is $\delta = 1$ Hz. The panels common parameters are $f_{\text{sampling}} = 10$ kHz and $\# \text{samples} = 1 \cdot 10^5$. The probe power for the three panels are $P_{\text{probe}} = (a) -126$ dBm (b) $-116$ dBm (c) $-106$ dBm.

Figure 4.27: Flux bias $\Phi_{dc} = 0.17 \Phi_0$. (a) Subharmonic oscillations generated through current driving. (b) Subharmonic oscillations generated through pumping the on-chip flux tuning line.
4.3. NONADIABATIC MODULATION OF A SQUID

![Figure 4.28: Subharmonic oscillation regions for (a) an external drive of the Josephson junction and (b) for pumping the Josephson junction through the on-chip flux pump line.](image)

To further investigate the effect of current driving versus flux pumping, a control sample was fabricated with a single Josephson junction instead of a SQUID, terminating the resonator. The Josephson junction is to first order flux insensitive and should, in the case of pure flux pumping, not give any effect. For the current drive it does not matter, whether the nonlinear element is a SQUID or a single junction, the effect should be the same since the SQUID is just effectively a flux-tunable junction. In Fig. 4.28, the two modulation methods are compared for the single junction sample. In (a) an external current drive signal is applied at $3\omega$, and subharmonic oscillations are measured at $\omega$. In (b) a microwave signal is applied to the on-chip flux pump line at $3\omega$ and again, even though the resonator is supposed to be insensitive to flux, subharmonic oscillations are measured at $\omega$. The similarity of the two panels in Fig. 4.28 indicate that there is some mechanism, coupling the flux pump signal to the resonator and cause a current drive. This coupling could be an effect of parasitic coupling between the flux pump line and the resonator. Another possible coupling mechanism would be via spurious modes in the sample box. From the available data there is no possibility to deduce which mechanism is the dominating one.

### 4.3.3 Period multiplication subharmonic oscillations

The subharmonic oscillations are not limited to period-tripling. In Paper B we show that we can generate, from period-doubling subharmonic oscillations, up to period-quintupling, in a $\lambda/4$-resonator. Both in Paper B, and here in this section of the thesis, I focus on SQUID modulation through the on-chip flux pump line. In the Paper we present how the period-multiplication effect in a superconducting resonator can be connected to the theory model presented by Guo et al. [78]. As discussed in the paper we have two cases, flux pumping at even or odd multiples of $\omega$. Pumping at even multiples correspond to parametric pumping, while pumping at odd multiples requires an asymmetry in the SQUID to generate a period-multiplication effect. It is not clear how large asymmetry is required but my measurements indicate that the imperfections in the fabrication process are enough. The expected asymmetry of a nominally symmetric SQUID is of the order of a
4. RESULTS

Table 4.2: An overview of different cases, if subharmonic oscillations are expected from theory or not for different modulation frequency multiples. The different SQUID modulation mechanisms are current driving and flux pumping. The SQUID can either be symmetric and have two junctions with equal critical current or asymmetric and have two junctions with different critical current. For the flux pump case there could be a parasitic coupling between the flux pump line and the resonator, this is in the table denoted CT for crosstalk.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Current drive</th>
<th>Flux pump</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Sym SQUID</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No CT</td>
</tr>
<tr>
<td>$2\omega$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$3\omega$</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$4\omega$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$5\omega$</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Few percent. A summary, of whether theory predicts existence of subharmonic oscillations or not for different cases, is found in Table 4.2. For current driving at even multiples, no effect is expected. Even so, I have measured period-doubling by current driving, see Appendix B.3.

Period-quadrupling subharmonic oscillations

Period-quadrupling subharmonic oscillations indicate that I use a modulation frequency that is four times the measurement frequency. An example of the effect, generated by applying a microwave signal at $4\omega$ to the flux pump line, is given in Fig. 4.29 for sample #4 biased at $\Phi \approx 0.1\Phi_0$. In panel (a) I present the subharmonic region and in the measurement points indicated by the white circles I extract the quadrature histograms presented in (b)-(g).

The histograms in Fig. 4.29 show quartet excited states. In some of the histograms there is an indication of an additional middle ground state apart from the quartet excited state, for instance panel (b,d,g). In panel (f), there are faint lines between neighbouring states, indicating that switching appears between neighbouring states. In contrast, the pattern in (b) indicates that switching also can go through the middle, because of the existence of the stable ground state.

One limiting factor here, visible in Fig. 4.29(a), is the limitation on pump power. The available microwave source could not supply more power. However, within the setup limitations I can conclude that for $4\omega$-pumping there is no detected frequency down-conversion for zero flux. Since the frequency down-conversion for pumping at even multiples is a regular parametric process, we would not expect any effect at zero flux.

As shown, a period-quadrupling effect is observed for a flux pumped tunable resonator. According to theory, no effect is expected for current driving. In my measurements I can show, within the limitations of the measurement setup, a minor effect at current driving. Though, the period-quadrupling effect is significantly stronger for flux pumping than for current driving, in agreement with theoretical arguments.
Figure 4.29: Period-quadrupling subharmonic oscillations generated by applying a microwave signal at frequency $4\omega$ to the flux pump line and measuring at $\omega$. (a) Region of subharmonic oscillations. Unfortunately, at the time, the measurement setup did not allow a larger pump power. (b)-(g) Quadrature histograms of the detected output signal. The measurement points are indicated by the white circles in (a).
Period-quintupling subharmonic oscillations

For generating period-quintupling subharmonic oscillations I apply a microwave signal to the flux pump line at a frequency $5\omega$ and I set the digitizer readout frequency to $\omega$. Following the discussion from Paper B, a generation of subharmonic oscillations at an odd multiple requires a small asymmetry in the SQUID. Alternatively, the pump signal can couple via a parasitic crosstalk from the flux pump line to the resonator modes and populate them. In turn this excitation can facilitate down-conversion to the first mode where I read out subharmonic oscillations. The data presented in Fig. 4.30 is extracted from sample #2. Panel (a) shows the generated output power and in (b)-(d) three quadrature histograms are presented, all measured in the strong output power part in the top left corner of (a).

To get an idea of how the system switches between the states a test with varying sampling frequency was performed, see Fig. 4.31. For the fastest sampling, panel (c), it seems like the switching is mainly through the middle state, possibly some nearest neighbour switching occurs as well. At the lower sampling rates, in panel (a) and (b), there are indications that there could be switching between neighbouring excited states.

Some reflections from the period-quintupling measurements. The required pump power to reach the subharmonic oscillation threshold is generally lower for period-quintupling
4.3. NONADIABATIC MODULATION OF A SQUID

Figure 4.31: Period-quintupling subharmonic oscillation histograms measured with different samplings frequencies, 10, 50 and 100 kHz. Here, the colour scale is presented in logarithmic scale to highlight the traces of switching between different states.

Figure 4.32: $6\omega$ subharmonic oscillations at the bias point $0.3\Phi_0$. In (a) the subharmonic oscillation region and in (b) a quadrature histogram extracted at the maximum generated output power in (a). Interestingly there seems to be only three states.

compared to period-quadrupling. Furthermore, the regions, in which the subharmonic oscillations are generated, are very similar for current driving and flux pumping in the period-quintupling case.

$6\omega$ subharmonic oscillations

In most samples, attempts at $6\omega$-modulation has not given any response. However, as a curiosity, for sample #2 an effect was found for modulation through the on-chip flux pump line, see Fig. 4.32. The quadrature histogram shows three bright spots not six. I show this only as a curiosity. For this sample, with a fundamental mode resonance frequency around 5 GHz, $6\omega \sim 30$ GHz. It is likely, that the SQUID plasma frequency is found lower or close to the modulation frequency. It remains unknown how the generated radiation is affected by direct excitation of the SQUID.
4. RESULTS

\[\delta_1/2\pi = (\omega - \omega_1)/2\pi \text{ [MHz]} \]

\[P_{\text{out}} \text{ [nW]} \]

\[P_p \text{ [dBm]} \]

\[P_{\text{out}} = \frac{\delta_1}{2\pi} (\omega - \omega_1)/2\pi \text{ [MHz]} \]

\[f_{\text{dig}} = 10.4 \text{GHz}, P_p = -12 \text{dBm}, f_s = 50 \text{kHz} \]

\[f_{\text{dig}} = 10.401 \text{GHz}, P_p = -12 \text{dBm}, f_s = 50 \text{kHz} \]

\[I \text{ [arb. units]} \]

\[Q \text{ [arb. units]} \]

\[\text{Counts}/10^3 \]

Figure 4.33: Parametric oscillation regions generated by flux pumping the SQUID at \(2\omega\), (a) 0.1 \(\Phi_0\) and (b) 0.3 \(\Phi_0\). Two typical quadrature histograms, (c) measured at the lower region boundary of (a) and (d) measured in the middle of the region in (a).

4.3.4 Standard parametric oscillator

For use as a comparison with the doubly tunable resonator, I here present some standard parametric oscillator properties, measured using the single-SQUID tunable resonator, sample #2 (see Table 4.1). Here we assume that the SQUID modulation mechanism is purely flux pumping. Remember the theoretical predictions in Fig. 2.23 and 2.24. The parametric oscillation region is, due to nonlinearity, shifted in frequency, an effect especially pronounced at low flux bias. In experiments this effect is easily observed, see Fig. 4.33(a) and (b). Furthermore, the system indeed shows a combination of a ground state and two \(\pi\)-shifted states at the lower oscillation boundary and two separate states in the middle of the region, see Fig. 4.33(c) and (d).

4.3.5 Photon generation in the doubly tunable resonator

In paper C our measurement results on the doubly tunable resonator are presented. As written, we find some very promising results in nice agreement with theory. For example, we find that, for a single pump measurement, the system behaves as a parametric...
oscillator, it forms a parametric region, and inside that region, a histogram can be measured illustrating two $\pi$-shifted states. Furthermore, for a double pump configuration, parametric regions can be measured, where the region threshold depends on the phase difference between the two pumps. However, there are also some contradicting findings and here I will show some additional measurement results, indicating parasitic effects. The experimental data in Paper C is measured on sample DTR #5. This is also the sample, from which the data in this section is collected.

For a single flux pump at $2\omega$, the doubly tunable resonator should behave very similar to the standard parametric oscillator. In Fig. 4.34 I have plotted the regions for (a) $\Phi_{dc,l} = \Phi_{dc,r} = 0.1 \Phi_0$ and (b) $\Phi_{dc,l} = \Phi_{dc,r} = 0.3 \Phi_0$, and these should be very similar to the regions of Fig. 4.33. It can be noted that the doubly tunable resonator generate very narrow parametric regions, compared to the standard parametric oscillator. In addition, the pump power required to generate an output signal is roughly 10 dB lower here compared to the standard parametric oscillator. Even though the on-chip flux pump lines and SQUIDs are very similar.

![Image of parametric regions](image-url)
4. RESULTS

Figure 4.36: Extracted thresholds for the parametric regions created by using both flux pumps in the doubly tunable resonator. The red line corresponds to a theory comparison using Eq. (2.72a).

Also for double pumping of the doubly tunable resonator, parametric regions can be generated. In Fig. 4.35, regions for pump phase differences 0, 90 and 180 degrees, for the bias point $\Phi_{\text{dc},1} = \Phi_{\text{dc},r} = 0.3 \Phi_0$ are presented. For each pump phase difference, the threshold has been extracted, see blue circles in Fig. 4.36. These thresholds have been compared to the model in Eq. (2.72a). The theoretical threshold can be extracted by setting $\epsilon = \sqrt{\delta^2 + \Gamma^2}$. For simplicity I work with $\delta \approx 0$. The effective pump amplitudes are given by

$$\delta f_{r,l} = M_{ac} \sqrt{2} \sqrt{\frac{P_p - P_{\text{att}}}{10}} / 50,$$

where $P_{\text{att}}$ denotes the attenuation on the pump line and $M_{ac}$ is the mutual inductance between the on-chip flux pump line and the SQUID. The resonator parameters used in the comparison are found in Table 2 of Paper C. The specific bias point used here has the resonance frequency $\omega_1 / 2\pi = 5.160$ GHz and $Q_{\text{tot}} = 1 \cdot 10^4$. The phase offset has been adjusted so that the minimum threshold is at zero. Since it is difficult to separate the attenuation and the mutual inductance I use the parameter $k = M_{ac} \sqrt{2} \sqrt{\frac{P_{\text{att}}}{10}} / 50$. I also assume, that the crosstalk is symmetric $\xi_{ac,r} = \xi_{ac,l} = \xi_{ac}$. The red theory comparison in Fig. 4.36 has been calculated for $k = 5.5 \cdot 10^{-5}$ and $\xi_{ac} = 1$. This theory trace does not fit perfectly to the data, by taking more imperfections into account such as differences in pump power for the two pumps, a better agreement could be found. However, the following paragraph will explain why there is no point in spending time finding a good fit using a flux pumping model.

To probe the microwave crosstalk between a SQUID and its opposite flux pump line, a doubly tunable resonator, where one SQUID was replaced by a continuous transmission line, was fabricated. Since there is only one SQUID, the effect of the opposite flux pump line is a direct measure of the ac crosstalk. In Fig. 4.37, single pump parametric regions for (a) using the flux pump line with a SQUID, and (b) using the flux pump line without a SQUID are presented. Ideally, there should be no effect when there is no SQUID, though as seen in the graphs, the generated effect is almost identical independently which flux
4.4. MICROWAVE MANIPULATION WITH THE TUNABLE COUPLING

![Graph showing pump power vs. detuning](image)

**Figure 4.37:** Single pump parametric regions for a single SQUID control sample. (a) Pumping on the flux pump line with a SQUID. (b) Pumping on the flux pump line without SQUID.

The pump line is used. This is an indication that the crosstalk is very close to 100% or the SQUID is modulated by another mechanism than flux pumping. A guess, supported by the measurement results of double pumping of the single SQUID sample (Fig. 4(c) in Paper C), is that the SQUID is current driven due to a parasitic coupling between the flux pump line and the resonator.

4.4 Microwave manipulation with the tunable coupling

Paper E and F present experimental results of microwave signal manipulation using the tunable coupling architecture in Fig. 2.29. There are two samples, A and B, where sample A is close to the critical regime and sample B is in the underdamped regime.

Paper E treats the measurement results of sample A. It is shown that the escape rate from the storage resonator to the transmission line can be tuned over three orders of magnitude, by changing the flux in the SQUID loop of the coupling resonator and thereby tune its resonance frequency. The storage resonator is loaded by tuning the coupling resonator on resonance with the storage resonator. Then the coupling is turned off for some time and then the microwave signal is released when the coupling is turned back on. In a controlled pulse sequence with varying storage time, a maximum storage time of 18 μs has been extracted. The possible storage time is mainly set by the intrinsic Q-value of the storage resonator, which for sample A is $6 \times 10^5$. This corresponds to a storage time of 18 μs for the storage resonator frequency 5.186 GHz. Note that there is an additional factor of $1/2$ to convert the voltage decay rate to an energy decay rate.

After storage the microwave signal is released by tuning the coupling resonator back on resonance. The fastest possible release is found for zero detuning between the coupling and storage resonator, 14 ns. However, the release time and the shape of the released signal can be controlled by choosing a finite detuning between the coupling and storage resonator.
In Paper F we perform experiments using sample B. This sample is in the underdamped regime, the coupling between the two resonators is stronger than the coupling to the transmission line. On resonance we observe oscillations in the two-mode system due to the relatively strong coupling between the resonators compared to the coupling to the transmission line. The energy has time to oscillate between the resonators before leaking out. This also means that, compared to sample A, the release time is much longer in sample B, limited by the coupling to the transmission line.

Direct loading of the storage resonator can only be done at the resonance frequency of the same. However, the coupling resonator can be tuned over a wide frequency range. If off-resonant, the coupling resonator can be loaded with a coherent state while leaving the storage resonator empty. Then by performing a swap operation, \textit{i.e.} tuning the resonators on-resonance, the contents of the two resonators can be swapped. The signal can be stored for some time in the storage and then a second swap operation is performed to transfer the signal back to the coupling resonator and read out. This process has been demonstrated in excellent agreement with theoretical predictions.
Conclusion and outlook

In this thesis I have presented three different systems that all rely on tunable superconducting resonators. These systems have been characterized and two of them, the subharmonic oscillations and the tunable coupling, have been shown very promising. Classically, we have shown interesting properties of these systems. Since low loss superconducting circuits are used, it is likely that quantum versions of the effects could also be created. Even though there are some problems with the doubly tunable resonator, it remains an interesting system, possible to use for experiments on relativistic effects, as well as for creation of non-classical states.

The period-tripling subharmonic oscillations are now extensively studied in the classical, weak signal regime. What remains unknown is for instance, the behaviour at higher signal levels, such as the double triplet state. There are also possibilities to further explore the negative anharmonicity regime. On a classical, many photon level the positive and negative anharmonicity results are not that different, however in the quantum regime, possibly there could be something interesting.

To further understand the more general period-multiplication subharmonic oscillation case, some more theory work is needed, in combination with more systematic measurements. Here, we show that the subharmonic oscillations exist for pumping at different multiples and there are reasons to believe that these could form interesting non-classical physics. However, from the present measurement results it is difficult to draw conclusions because of several unknowns, one is the possible presence of direct SQUID excitations, due to the proximity of the SQUID plasma frequency. Therefore, new measurements should be done with a SQUID where the plasma frequency is maximized and possibly the resonator frequency minimized, with respect to the limitations of our measurement setup.

The doubly tunable resonator has been realized in this thesis, and remains an interesting idea even if there are technical problems. I strongly believe that the parasitic coupling problem causing huge crosstalk and current driving could be solved. A stepped impedance design could be one way to go, to increase the distance between the pump frequencies and higher resonator modes. Another solution could be to change the sample design, the impedance in the ground plane loop, formed by the resonator centre conductor and the ground plane, could be increased, to avoid circulating currents. Possibly, the best design
is a variation of the parallel on-chip flux tuning line design (Fig. 4.10(b)). Since this one has an outlook to cut at least partly also the ac crosstalk.

At the moment, no attempts have been made to measure the time dilation in the doubly tunable resonator. However, the problems encountered in the parametric pumping are not necessarily relevant for the twin paradox measurements. Assuming that the parasitic coupling between the on-chip flux tuning lines and the resonator, is connected to resonator modes, the space trip could be realized with signals with frequencies far from the mode frequencies.

In both the subharmonic oscillations project and the doubly tunable resonator there is potential interaction with parasitic box or other background modes. Both these projects would benefit from having less parasitic modes in the sample boxes. This could be achieved, both by decreasing the frequency, since there are less parasitic modes at lower frequencies, or by designing new better boxes where the parasitic modes are pushed to higher frequencies.

The tunable coupling is shown to work very well for storage of microwave fields. There is an excellent agreement between theory and experiments. One improvement would be to increase the Q-value of the storage resonator, to increase the possible storage time, with the new fabrication recipe (section A.2) colleagues in the group has shown Q-values of plain resonators of over a million [79]. Another interesting next step would be to incorporate this tunable coupling in a larger quantum network for instance as a tunable coupling between quantum bits.
Appendix A

Cleanroom recipies

A.1 Nb-recipe, sapphire substrate

1. Annealing sapphire substrates
   - 1050°C for 6 h

2. Nb-resonator
   - Sputter Nb, 80 nm
   - Spin resist: S1805, 3000 rpm, 1500 rpm, 45 s
   - Bake 2 min 115°C
   - Exposure, pneumatic focus, Focus:−5 %, Intensity:80 %, Transmission:100 %
   - Develop 40 s in MF319
   - Rinse in water
   - Blowdry
   - Ashing 30 s, 50 W
   - Reactive Ion Etch with NF3 (50 sccm), laser endpoint detection, around 60 s
   - Ashing 2 min, 50 W, to remove the top layer of the resist.
   - Clean in warm remover ~ 10 min
   - Rinse in IPA
   - Blowdry

3. Au-structures
   - Ashing 2 min, 50 W
   - Spin bottom resist: LOR3A 3000 rpm, 1500 rpm, 45 s
   - Bake 5 min 190°C
- Spin top resist: S1813 3000 rpm, 1500 rpm, 45 s
- Bake 2 min 110°C
- Laser writer exposure, pneumatic focus, Focus:0 %, Intensity:100 %, Transmission:100 %
- Development 60 s in MF319
- Rinse in water
- Blowdry
- Ashing 15 s, 50 W
- Ar-ion etching, 6 min, 300 V, 1 A
- Evaporate, Cr: 2 nm, Au: 110 nm, Pd: 10 nm
- Lift-off in remover, needs a full day or over night
- Rinse in IPA
- Blowdry

4. Lithography, Al-layer

- Ashing 2 min, 50 W
- Spin bottom resist: MMA(8.5)EL10, first step, 500 rpm, 333 rpm, 6 s, second step, 2000 rpm, 400 rpm, 50 s
- Bake 5 min, 170°C
- Spin top resist: ARP6200/2 2:1 500 rpm, 333 rpm, 6 s
- Bake 5 min 160°C
- E-beam exposure, aperture 6, 2 nA, dose: 380 µC/cm²

5. Dicing, without any extra protective layer.

6. Deposition Al-SQUID, two-angle evaporation

- Develop top layer, n-Amylacetate 1 min
- Blowdry
- Develop bottom layer, MIBK/IPA, 5 min (time has to be adjusted for each wafer)
- Quick rinse in IPA
- Blowdry
- Ashing 15 s, 50 W
- Ar-ion etching, two angles ±30°, 1+1 min
- Evaporate, two angles with oxidation in between, ±30°, 30 + 110 nm, dynamic oxidation 0.2 mbar 30 min
- Lift-off in warm remover, important to add some movement otherwise the Al can collapse onto the surface and stick (see Fig. C.2).
- Clean in IPA
- Blowdry

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A.2 Al-recipe, high resistivity silicon substrate

1. Cleaning of high resistivity Silicon
   - Acetone 5 min with ultrasonic 100% normal mode
   - Methanol 5 min with ultrasonic 100% normal mode
   - IPA 5 min with ultrasonic 100% normal mode
   - Standard cleaning 1, 10 min
   - Quick dump rinse (QDR)
   - HF 2% 1 min
   - QDR
   - Blowdry

2. Deposition of aluminum in Plassys
   - Heat to 300°C and keep for 10 min
   - Cool down and pump overnight, pressure = 6e − 8 mbar
   - Deposit 150 nm at 1 nm/s
   - Static oxidation 10 mbar, 500 sccm, 10 min

3. Etching of aluminium
   - Spin AZ1512HS at 4000 rpm, 1 min
   - Bake 100°C 50 s
   - Expose in DWL (laser writer), F: -5, I: 80, T: 100
   - Develop AZ Developer:H₂O 1:1 30 s
   - QDR
   - Blowdry
   - Ashing 20 s, 25 W
   - Etch in transene type A 40°C 1 min 5 s
   - QDR
   - Blowdry
   - 1165 85°C 5 min + 5 min in ultrasonic 40% sweep mode
   - Methanol with ultrasonic 40% sweep mode
   - IPA with ultrasonic 40% sweep mode
   - Blowdry

4. Deep etching of aluminium and silicon
   - Spin AZ1512HS at 4000 rpm 1 min
• Bake 100°C 50 s
• Expose in DWL, F: −5 %, I: 80 %, T: 100 %
• Develop AZ Developer:H₂O 1:1 30 s
• QDR
• Blowdry
• Ashing 20 s, 25 W
• Etch in transene type A 40°C 1 min 10 s
• QDR
• Blowdry
• Reactive Ion Etch with NF₃
• Ashing 30 s, 50 W
• 1165 85°C 5 min + 5 min in ultrasonic 40% sweep mode
• Methanol with ultrasonic 40% sweep mode
• IPA with ultrasonic 40% sweep mode
• Blowdry

5. SQUIDs, cross design

• Spin Copolymer EL10 3000 rpm 1 min
• Bake 160°C 5 min
• Spin PMMA A6 6000 rpm 1 min
• Bake 160°C 5 min
• Expose with PEC (ebeam), 2 nA, 800 μC/cm²
• Develop MIBK:IPA 1:3 90 s
• IPA 10 s
• Blowdry
• Ashing 20 s, 25 W
• Load in the Plassys evaporator, align and note planetary angles
• Deposit 50 nm 1 nm/s at 45° tilt
• Static oxidation 2 mbar 100 sccm 15 min
• Rotate stage
• Deposit 110 nm 1 nm/s at 45° tilt
• Static oxidation 10 mbar 500 sccm 10 min
• Lift off in 1165 85°C
• Fresh 1165 with ultrasonic 40% sweep mode
• Methanol with ultrasonic 40% sweep mode
• IPA with ultrasonic 40% sweep mode
• Blowdry

6. Plasters

• Spin Copolymer EL10 3000 rpm 1 min
• Bake 160°C 5 min
• Spin PMMA A6 6000 rpm 1 min
• Bake 160°C 5 min
• Expose without PEC (ebeam), 35 nA, 700 µC/cm²
• Develop MIBK:IPA 1:1 90 s
• IPA 10 s
• Blowdry
• Ashing 20 s, 25 W
• Ion mill 400 V, 20 mA, 60 V, 3.5 min
• Deposit 200 nm 1 nm/s
• Static oxidation 10 mbar 500 sccm 10 min
• Lift off in 1165 85°C
• Fresh 1165 with ultrasonic 40% sweep mode
• Methanol with ultrasonic 40% sweep mode
• IPA with ultrasonic 40% sweep mode
• Blowdry

7. Cleave into chips and cleaning

• Spin S1813 3000 rpm 1 min
• Bake 110°C 2 min
• Dice in Disco DAD
• 1165 85°C for 5 min on hotplate and then 5 min in ultrasonic bath 40% sweep mode
• Methanol 5 min with ultrasonic 40% sweep mode
• IPA 5 min with ultrasonic 40% sweep mode
• Blowdry
• Ozone 30 min
A.3 Al-recipe, silicon substrate, lift-off recipe

1. Cleaning
   - 10 – 15 min, remover 1165, 60°C
   - Ultrasonic bath, 100%, 1 min
   - IPA bath, 2 min circulation
   - QDR bath
   - Blowdry
   - Ashing 1 min, 50 W

2. Au-contacts
   - Spin HDMS primer. 3000 rpm, 1 min, \( t_{acc} = 1.5 \) s bake 1 min 110°C
   - Spin LOR3B, 3000 rpm, 1 min, \( t_{acc} = 1.5 \) s bake 5 min 200°C
   - Spin S1813, 3000 rpm, 1 min, \( t_{acc} = 1.5 \) s bake 2 min 110°C
   - Photolithography, exposure 8.5 s, low-vac mode
   - Development 40 s, MF319 (lift up after 15 s)
   - QDR bath+blowdry
   - Deposition, Ti: 30 Å (2 Å/s) Au: 800 Å (5 Å/s) Pd: 100 Å (2 Å/s)
   - Remover 1165, 70°C, 1 h maybe less
   - IPA bath, 2 min circulation
   - QDR bath+blowdry

3. Semi-dicing (cut halfway through the wafer)
   - Spin S1813, 3000 rpm, 1 min, 1.5 s bake 110°C 3 min
   - Dicing in the Loadpoint. HUB blade 50 μm, tape thickness 75 μm, wafer thickness 380 μm, depth of cut 0.950 mm, semidicing 0.725 mm
   - 1165, 70°C, rinse with IPA+blowdry
   - Ashing 1 min, 50 W

4. SQUIDs, exposure done on full wafer development and evaporation on single chips
   - Spin MMA(8.5)EL10, step one 500 rpm, 5 s, \( t_{acc} = 2 \) s, step two 2000 rpm, 45 s, \( t_{acc} = 5 \) s, bake 5 min, 170°C
   - Spin ZEP520A 1:1 anisole, 3000 rpm, 45 s, \( t_{acc} = 0.5 \) s, bake 5 min 170°C
   - Exposure (ebeam)
   - Develop top layer, o-xylene 2 min
   - IPA rinsing+blowdry
   - Develop bottom layer, H2O IPA 1:4, 5 min 30 s
• Rinse in IPA
• Blowdry gently!
• Evaporation (two-angle) Plassys, 45 nm+55 nm, oxidation at 0.2 mbar, 30 min, angle ±26.5°
• Remover 1165, 20 min ”move” (to get rid of sticking Al) 70°C
• Rinse in IPA

A.4 Nb-recipe, silicon substrate

1. Cleaning
   • Dip in HF bath 30 s to etch oxide
   • QDR bath

2. Sputtering, DCA
   • Load the wafer within 20 min after HF cleaning!
   • Annealing 700°C, 20 min
   • Sputter Nb, 80 nm

3. Lithography
   • Spin resist: UV60-0.75, 3000 rpm, \( t_{acc} = 1.5 \text{s}, 1 \text{ min} \), bake 2 min 130°C
   • E-beam exposure, aperture 8, 45 nA, dose: 34
   • Bake 90 s 130°C
   • Develop 55 s MF24A

4. Etching
   • Reactive ion etch with NF\(_3\) plasma (50 sccm), laser endpoint detection, around 40 s
   • To remove redeposited polymer layer, ashing 2 min, 50 W
   • Clean in remover

5. Lithography, Au-layer
   • Spin bottom resist: MMA(8.5)EL10, step one 500 rpm, 5 s, \( t_{acc} = 2 \text{s} \), step two 2000 rpm, 45 s, \( t_{acc} = 5 \text{s} \), bake 5 min, 170°C
   • Spin top resist: ARP6200/2 2:1, 3000 rpm, 1 min, \( t_{acc} = 1 \text{s} \), bake 5 min 160°C
   • E-beam exposure, aperture 8, 70 nA, dose: 280 \( \mu \text{C/cm}^2 \)
   • Develop top layer, n-Amylacetate 2 min
   • Blowdry
6. Deposition Au-layer
   - Ashing 15 s, 50 W
   - Ar-ion etching, 7 min, 300 V, 1 A
   - Evaporate, Cr: 20 Å, Au: 1100 Å, Pd: 100 Å
   - Lift-off in remover

7. Dice alignment cuts
   - Spin protective resist. S1813, 2000 rpm, 1 min, 1.5 s bake 110°C 3 min
   - Use HUB-blade 50 μm
   - Cut one chip size out from pattern
   - Remove protective resist with remover, rinse with IPA+blowdry

8. Lithography, Al-layer
   - Spin bottom resist: MMA(8.5)EL10 500 rpm, 5 s, $t_{acc} = 2$ s 2000 rpm, 45 s, $t_{acc} = 5$ s bake 5 min, 170°C
   - Spin top resist: ARP6200/2 2:1 3000 rpm, 1 min, $t_{acc} = 1$ s bake 5 min 160°C
   - E-beam exposure, aperture 6, 2 nA, dose: 300 μC/cm²

9. Dicing
   - Use the same blade as for alignment dicing
   - Dice the wafer into single chips
   - Carefully remove the chips from the tape

10. Deposition Al-layer
    - Develop top layer, n-Amylacetate 2 min
    - Blowdry
    - Develop bottom layer, H2O/IPA 1:4, 7 min
    - Quick rinse in IPA
    - Blowdry
    - Ashing 15 s, 50 W
    - Ar-ion etching, two angles ±30°, 1+1 min
    - Evaporate, two angles with oxidation in between, ±30°, 30 + 110 nm, dynamic oxidation 0.2 mbar 15 min
    - Lift-off in remover
Appendix B

Various interesting measurement results

Over the past years I gathered a huge amount of measurement data. Here I show some results that did not make it into the main thesis but make nice figures.

B.1 DC-tuning with parasitic superconducting loops

![Diagram](image.png)

Figure B.1: A SQUID-terminated $\lambda/4$-resonator. On the horizontal axis I step the static flux. Instead of a single parabola showing the tuned resonance frequency I found a parabola with a lot of fine structure. Nice picture but not optimal for the experiments I want to do. A further zoom in shows even more fine structures.

A very interesting effect I have found was the dc tuning of Fig. B.1. This pattern is measured in a typical $\lambda/4$-resonator with a SQUID at the grounded end. The sample
was fabricated using recipe A.1, niobium on sapphire with aluminium SQUIDs. However, what was found is that in the aluminium evaporation, due to the two angle evaporation metal was evaporated on the edges of the chip. Since there is no resist on the edges of the chip this metal was not removed by the lift-off. In conclusion, I had a sample with an additional and big superconducting loop. This superconducting loop also had a weak link, the thin aluminium on the chip edge. I believe that the fine structure in Fig. B.1 was created by fluxes tunnelling in and out of this additional big superconducting loop.

B.2 Stable and unstable resonance

In section 3.4.1 I discuss different aspects of cutting ground planes into pieces versus keeping them continuous and risk trapping magnetic flux in large superconducting loops. In Fig. B.2 I show test measurements of two samples, the design difference between the two is that in (a) all possible superconducting flux traps are cut but in (b) flux can be trapped in big superconducting loops, similar to the drawing in Fig. 3.14.

These two samples and measurement results show two things. First it can be noted that the flux period is much smaller in (a) than in (b). This can be understood by the fact that when keeping the continuous groundplane and form superconducting loops, these loops expel magnetic fields. Another observation is that the extracted resonance frequencies in (b) show a very noisy pattern. In contrast the frequency pattern in (a) looks very stable. My interpretation of this is that the big superconducting loops cause flux trapping and noise.
Figure B.3: Period doubling subharmonic oscillations generated by current driving the SQUID in a $\lambda/4$-resonator. (a) Subharmonic oscillation region. (b) Histogram measured on the lower boundary of the region. (c) Histogram measured in the middle of the region.

### B.3 Period doubling subharmonic oscillations

Figure B.3(a) shows a period doubling subharmonic oscillation region, generated by current driving. Panel (b) and (c) show two quadrature histograms. In resemblance with the pattern for period tripling subharmonic oscillations, the low power boundary of the region shows an additional ground state in the middle. In the middle of the region, the histogram only presents two excited states. This is similar to the theoretical prediction for a parametric oscillator, Fig. 2.24.

The interesting part is that these subharmonic oscillations should not exist. According to theory (table 4.2) only current driving at odd multiples of the resonance frequency should give frequency down-conversion. However, possibly that if we just apply a strong enough drive signal higher order effects occur and period doubling can be observed. It can be noted that the threshold for generating period doubling through current driving is significantly higher than for generating period tripling.
Appendix C

Various fabrication results

Here follows some figures with varying fabrication results in the cleanroom.

C.1 Silicon substrate

When Mathieu Pierre and myself started fabricate samples with niobium on silicon we had some problems with the etching, this is shown in Fig. C.1. The problem was solved by switching the etch substance from CF$_4$ to NF$_3$. In Fig. C.2 I show some results from SQUID fabrication.

Figure C.1: (a) Niobium on silicon, physically etched by CF$_4$. The grains on the substrate could be redeposition of some carbon compound, or something else. The partly transparent thin film could be redeposition of etch waste on the resist walls that does not lift off with the resist. The problem with the grains was solved by changing the etching substance to NF$_3$. (b) A Josephson junction on a substrate covered with grains (see Fig. C.1). It seems like the grainy structure on the substrate makes the aluminium to form a grainy film. The black shadows could be bad cleaning of the resist, possibly remaining remover or similar.
Figure C.2: (a) A Josephson junction where the resist bridge has got some damage and caused a waist structure of the junction. This sample is fabricated using recipe A.4. The resonator is made of niobium and then the SQUIDs in aluminium. To ensure good contact between the niobium and aluminium, the niobium oxide is argon-ion-etched. My guess is that the waist structure of the junction is due to either damage from the ion-etch or by violent development. (b) Lift-off problems of the SQUID loop. If lift-off is left unattended the aluminium, supposed to be lifted off, can collapse to the chip surface and stick there. If instead, the lift-off is active and there is some movement in the resist the aluminium is properly lifted. In this case the substrate was silicon.

Figure C.3: Niobium on silicon, after etching and cleaning. In both pictures the grey areas correspond to silicon and the other is niobium. (a) There is something that looks like droplets on the substrate. Unclear what. (b) It looks like the metal has been flowing out.

C.2 Sapphire ebeam exposure results

Sapphire is an insulating substrate which makes it very sensitive to charging effects. In Fig. C.5, I show two examples of SQUIDs in the making, the pictures are from after ebeam exposure and development. This first example shows a working exposure. In Fig. C.6 I show a second set of pictures illustrating charging effects from an exposure that failed due to lack of grounding path.
Figure C.4: Zoom in on a niobium surface.

Figure C.5: SQUIDs in the making. This is typically how my resist patter looks after development. Note the nice undercut and that these samples after aluminium evaporation form working SQUIDs. However, apart from the yellowish undercut there is also a more uneven blueish edges. The origin of these uneven edges is unknown but possibly it has to do with charging effects due to the insulating sapphire.
Figure C.6: This figure shows the results of exposing the SQUIDs on a sapphire wafer, where the Nb has been etched so that each individual chip forms a Nb island. (a) Narrow Nb strip between the resonator and the on-chip flux pump line. I believe this is a charging/heating effect due to current concentration in a narrow strip. (b) SQUID test structure after development. Here, the exposed structure is displaced from its designed position and also underexposed. Probably, due to charging effects, the current beam is slightly misaligned.

Figure C.7: Due to the insulating substrate, SEM imaging on sapphire samples is more than difficult than silicon samples. Here I show an example of how a Josephson junction fabricated on a sapphire substrate. The dark shadows that makes the metal look dirty is Espacer, a chemical spun on top of the sample to help with charge distribution.
Appendix D

Technical aspects of the subharmonic oscillation setup

D.1 Histograms with phase drift

Figure D.1: To the left, normal histograms. To the right, histograms measured with phase drift.

Figure D.1 illustrates what happens when the measurement frequency of the digitizer do not match 1/3 of the drive frequency. The measurements are done by driving the SQUID in a λ/4-resonator at 3ω and measuring at ω. In the two left panels the detection frequency of the digitizer is exactly 1/3 of the signal generator drive frequency. In the two right panels the digitizer frequency, due to limited frequency resolution on the instrument...
and that not all numbers are possible to divide by three, has a small mismatch with 1/3 of the drive frequency.

### D.2 Compare a setup with circulators and one with directional coupler

Even though microwave components are rated for a specific frequency range it is usually possible to get a signal through also at a higher frequency. However, there could be extra attenuation, noise and nonlinearity in the response. In the first current driving measurements at 3ω, the setup was not planned with current driving in mind. Therefore, the drive signal was sent through circulators. It worked but with extra attenuation and background ripples. The difference between a measurement with a circulator and a directional coupler setup is illustrated in Fig. D.2. Both measurements are done on exactly the same sample and at the same bias point.

Figure D.2: A comparison between a subharmonic oscillation region measured with a (a) directional coupler setup and (b) a circulator setup.
References


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Appended Papers
Period-tripling subharmonic oscillations in a driven superconducting resonator
Period-tripling subharmonic oscillations in a driven superconducting resonator

Ida-Maria Svensson,* Andreas Bengtsson, Philip Krantz, Jonas Bylander, Vitaly Shumeiko, and Per Delsing

Microtechnology and Nanoscience, Chalmers University of Technology, SE-41296 Göteborg, Sweden

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We have observed period-tripling subharmonic oscillations in a driven superconducting coplanar waveguide resonator operated in the quantum regime, $k_B T \ll \hbar \omega$. The resonator is terminated by a tunable inductance that provides a Kerr-type nonlinearity. We detected the output field quadratures at frequencies near the fundamental mode, $\omega/2\pi \sim 5\, \text{GHz}$, when driving the resonator with a current at $3\omega$, with amplitude exceeding an instability threshold. We observed three stable radiative states with equal amplitudes, phase shifted by $2\pi/3$ rad, red detuned from the fundamental mode. The down-conversion from $3\omega$ to $\omega$ is strongly enhanced by near-resonant excitation of the second mode of the resonator and the cross-Kerr effect. Our experimental results are in quantitative agreement with a model for the driven dynamics of two coupled modes.

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I. INTRODUCTION

Nonlinear dynamical systems exhibit a vast variety of behaviors, from simple effects such as harmonic generation to sophisticated multiple bifurcations to pattern formation and chaos [1–3]. Particularly interesting are strongly nonlinear phenomena in the quantum regime, which can be realized in low-dissipation microwave systems such as circuit quantum electrodynamics (cQED) devices. Such phenomena play a central role and are widely employed in quantum information technology for qubit readout [4–6], photon entanglement [7–10], and generation of Schrödinger cat states [11,12].

Period-multiplying subharmonic oscillations [13,14] constitute a particular class of nonlinear phenomena. The oscillations appear as a nonlinear response at the oscillator frequency to an external drive at a multiple of the resonant frequency. In the quantum picture, the elementary process that underlies the subharmonic oscillations is a decay of a single photon into three, four, or more photons. The subharmonic oscillations are described by nonperturbative solutions to the dynamical equations, which appear abruptly and coexist with the stable vacuum state. In this respect the subharmonic oscillations distinctly differ from conventional parametric oscillations, which gradually emerge as a result of the vacuum instability. This difference is analogous to the difference between a first- and a second-order phase transition [15]. Furthermore, a symmetry-breaking aspect of this difference has important implications for the quantum dynamics of the period-tripling oscillations [16].

Although the period-multiplying phenomenon is theoretically explained in textbooks, experimental demonstrations are not common. A few early observations of subharmonic resonances in electromagnetic devices were performed on essentially classical electrical circuits with saturable inductors [17] or varactors [18]. More recent reports concern subharmonic resonances in lasers [19,20]. In Josephson circuits, the period-multiplying phenomenon has not received much attention; instead, research was focused on transition to chaos [21,22] and, lately, on bifurcation phenomena [23,24] and parametric oscillations [5,6,25–27]. Only recently have subharmonic oscillations in the quantum regime been theoretically discussed in the context of cQED [16,28,29].

In this paper we report the experimental observation of period-tripling subharmonic oscillations in a driven superconducting resonator in the quantum regime, $k_B T \ll \hbar \omega$ [30–32], where the thermal energy is much smaller than the energy of a single photon. We drive the nonlinear resonator with a harmonic signal with power $P_d$ at frequency $3\omega$, approximately equal to three times the fundamental resonator mode frequency, and observe a strong response at $\omega$. The output microwave signal consists of three correlated beams with equal amplitudes and different phases, shifted by $2\pi/3$ rad. The oscillations are detected within a certain window of the driving field amplitude: they start at finite-threshold detuning within the resonator bandwidth and persist deep into the red-detuning region.

Our observations can be qualitatively understood from the theory of a nonlinear oscillator [13,14]. When a driving force with off-resonant frequency $3\omega$ is applied, it generates a linear response at the same frequency, which is down-converted to frequency $\omega$ due to nonlinearity. The down-conversion has the highest efficiency when the detuning, $\delta_1 = \omega - \omega_1$, from the oscillator resonant frequency $\omega_1$ is small, $\delta_1 \ll \omega_1$.

However, application of this single-mode scenario to the resonator setting requires additional considerations. In our experiment, we drive the resonator close to its second mode, $\omega_2$, such that $\delta_2 = 3\omega - \omega_2 \ll \omega_1$. As a result, the response at the driving frequency becomes strongly enhanced and nonlinear, and the system dynamics is well described by two strongly interacting modes. This situation is different from the single-mode oscillator model explored in previous works on subharmonic oscillations. In fact, this resonant enhancement of the external drive, by more than three orders of magnitude, is crucial for the possibility to access the subharmonic oscillation regime in experiments.

II. EXPERIMENT

To observe subharmonic oscillations we use frequency-tunable coplanar waveguide microwave resonators [30–32]. The resonator is capacitively coupled to a 50 $\Omega$ transmission line on one end and grounded via a superconducting quantum
Chip
Cryostat
300 μm
15 μm
DC flux bias
AC flux mod.

FIG. 1. (a) Optical micrograph of one of the samples. The four coplanar waveguide resonators meander between the on-chip flux line on the left and the contact pads on the right. (b) Zoom of the SQUID that terminates the bottom resonator. The SQUID is designed with two identical Josephson junctions. (c) Measurement setup. Microwave signals are applied via attenuated coaxial cables, one for direct driving by an external ac current and one for flux modulation. To separate input and output, a directional coupler is used to route the signal. A static magnetic flux is induced by a superconducting coil. The output signal is amplified in a 4–8-GHz bandwidth by a cryogenic amplifier as well as a room-temperature amplifier. The quadrature voltages are acquired by heterodyne detection followed by digital demodulation.

The measurement setup is sketched in Fig. 1(c). For static magnetic flux biasing of the SQUID we use a superconducting coil mounted close to the sample box. The SQUID nonlinearity can be modulated by applying a microwave signal as an external drive of the current through the SQUID $I_s$ or by flux modulation. We focus mainly on the external driving. The resonator output signal is amplified by a low-noise cryogenic amplifier at room temperature, before being sampled by a digitizer. To maintain phase coherence, a 10-MHz signal is used to lock the signal generator and the digitizer together. The digitizer down-converts both the in- and out-of-phase quadratures, $I(t)$ and $Q(t)$, with a local oscillator before digitizing the data at an effective sampling rate $f_s$ during a time $t_f$. From the individual quadratures the total output power after amplification can be calculated as $P_{tot} = \langle I^2 \rangle + \langle Q^2 \rangle$.

We directly probe the first resonator mode in a reflection measurement. Higher modes can be measured only indirectly due to the 4–8-GHz bandwidth limitation of our setup. To detect the second mode we use parametric up-conversion \cite{33,34}: we modulate the magnetic flux penetrating the SQUID loop at the difference frequency of the first and the second modes, $\omega_2 - \omega_1$, while simultaneously applying a weak drive tone at $\omega_1$. The flux pump converts photons from the first to the second mode, resulting in an avoided crossing, as shown in Fig. 2, from which we can determine the difference frequency. In Fig. 3(a) we present the extracted frequencies of the lowest mode for our sample together with a fit to the spectral dispersion, Eq. (A1). The extracted parameters are presented in Table I. The resonator is overcoupled and has a narrow bandwidth, less than 1 MHz.

III. SUBHARMONIC OSCILLATIONS

A. Observations

We observe period-tripling subharmonic oscillations at $\omega \approx \omega_1$ by applying an external signal at $3\omega$. This effect has been observed in several samples, although in this paper we present data for only one sample. As can be seen in Fig. 4(a), we observe subharmonic oscillations in a range of red detuning, $\delta_1 = \omega - \omega_1 < 0$, and for signal generator drive powers above $-13$ dBm. This range of drive powers and detunings forms region II, where the subharmonic oscillations are visible. Above and below, in regions I and III, respectively, no subharmonic oscillations are observed.
We also investigated the quadratures of the oscillations using histograms of the $I(t)$ and $Q(t)$ signals. In Fig. 4(c) we show a background histogram (peaked at $I = Q = 0$) illustrating the system noise level. This histogram represents the system ground state. At higher drive power, inside region II, the histograms feature three well-defined stable states forming a regular triangle [see Figs. 4(e) and 4(f)]. At the low-power edge of region II the system shows four states [see Fig. 4(d)]. This observation is in full agreement with the phase portrait of the subharmonic oscillator (see Fig. 7) featuring four coexisting stable states; the silent ground state and the three excited states.

The system switching rates between the states are different for different operating points, $\delta_1$ and $P_d$. Analysis of the underlying time-domain data yields a switching rate of 1 kHz in Fig. 4(d) and 15 kHz in Fig. 4(f). In Figs. 4(d) and 4(e) the histograms show clearly separated states, while in Fig. 4(f) the states are connected by faint lines. These lines indicate enhancement of the stochastic switching between the steady states. When the system switching rate becomes comparable to the sampling rate, $f_{\text{fs}} \sim f_s$, the states are averaged together.

Lowering the sampling rate to 10 kHz makes the switching processes between the stationary states more visible (see Fig. 5). In region II the transitions occur only between the excited states, forming a triangle configuration, while at the border of regions II and III the transitions connect the ground and excited states, forming a star configuration.

### B. Theory

To explain the experimental observations and establish a basis for quantitative comparison, we perform a theoretical analysis based on the theory for two-mode resonant dynamics in a frequency-tunable resonator [26, 34]. The two-mode equations for slowly varying Heisenberg operators of the coupled modes, $a_1$ and $a_2$, in the doubly rotating frame with frequencies $\omega$ and $3\omega$ have the form

\[
i\dot{a}_1 + (\delta_1 + i\Gamma_1) + \alpha_1 a_1^\dagger a_1 + 2\alpha a_2^\dagger a_2) a_1 + \tilde{\alpha} a_1^3 a_2 = 0,
\]

\[
i\dot{a}_2 + (\delta_2 + i\Gamma_2) + \alpha_2 a_2^\dagger a_2 + 2\alpha a_1^\dagger a_1 a_2 + \frac{\tilde{\alpha}}{3} a_1^3
\]

\[
= \sqrt{2\Gamma_{2,\text{ext}}} B_2.
\]

Here, the amplitude of the fundamental mode, $a_1$, describes the subharmonic oscillator, while that of the second mode, $a_2$, acts as an effective parametric pump. $B_2$ is the complex amplitude of the external drive and $\Gamma_2$ is the mode damping.

Explicit equations for the external damping $\Gamma_{2,\text{ext}}$ and the Kerr coefficients $\alpha_n$ are presented in Eqs. (A5) and (A6). The cross-Kerr coefficients are related to the Kerr coefficients, $\alpha = \sqrt{\alpha_1\alpha_2}$ and $\tilde{\alpha} = \sqrt{\alpha_1^3}\alpha_2$.

Equations (1) are associated with and can be derived from the quantum Hamiltonian,

\[
H / h = -\sum_{n=1,2} \left( \delta_n a_n^\dagger a_n + \frac{\alpha_n}{2} a_n^\dagger a_n^2 \right) - 2\alpha a_1^\dagger a_2^\dagger a_2 - \frac{\tilde{\alpha}}{3} (a_1^\dagger a_2 + a_2^\dagger a_1) + \sqrt{2\Gamma_{2,\text{ext}}} (B_2 a_2^\dagger + B_2^* a_2).
\]

Subharmonic oscillations are essentially a classical phenomenon since a large number of photons is generated in the resonator. Thus we restrict the analysis to the quasiclassical solutions of Eq. (1), neglecting quantum effects. A detailed description of the eigenfunctions and tunneling rates can be found in Ref. [16] for the Hamiltonian (2) in the single-mode case.

The trivial quasiclassical solution to Eq. (1), $a_1 = 0$, describes a silent oscillator state. It is always stable [see the Appendix, Eq. (A3)]. The nontrivial solutions describing stable steady states of the excited oscillator consist of a phase-degenerate triad, with the states being stable within the region of existence, Eq. (A14). In terms of a polar parametrization of

### Table I. Resonator and SQUID parameters: $d$, resonator length; $L_c$, SQUID critical current; $C_s$, SQUID capacitance; $\gamma_0 = L_{0,\text{crit}}/L_{c,\text{crit}}$ the inductive participation ratio; $L_0$ and $C_0$, inductance and capacitance per unit length; $\omega_0(0)$, resonant frequency of the first mode at $\Phi = 0$; $3\omega_0(0) - \omega_2(0)$, spectrum anharmonicity; $2\Gamma_{1,0}$, damping of the first resonator mode; $Q_1,\text{int}$ and $Q_1,\text{ext}$, quality factors of the fundamental mode.

<table>
<thead>
<tr>
<th>$d$ (μm)</th>
<th>$L_c$ (μA)</th>
<th>$C_s$ (pF)</th>
<th>$\gamma_0$ (%)</th>
<th>$L_0$ (μH/m)</th>
<th>$C_0$ (nF/m)</th>
<th>$\omega_0(0)/2\pi$ (GHz)</th>
<th>$[3\omega_0(0) - \omega_2(0)]/2\pi$ (MHz)</th>
<th>$2\Gamma_{1,0}(0)/2\pi$ (MHz)</th>
<th>$Q_1,\text{int}(0)$</th>
<th>$Q_1,\text{ext}(0)$</th>
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</thead>
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<td>5080</td>
<td>1.90</td>
<td>86.1</td>
<td>7.7</td>
<td>0.44</td>
<td>0.16</td>
<td>5.504</td>
<td>136</td>
<td>0.38</td>
<td>61.1 × 10^3</td>
<td>19 × 10^3</td>
</tr>
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</table>
FIG. 4. (a) Intensity of the subharmonic oscillation output signal as a function of drive power and detuning for $\Phi = 0$. The oscillations are detected in region II. The solid white line corresponds to the maximum output signal for each measurement frequency; the white star at the end of the solid white line represents the subharmonic oscillation frequency threshold. The dashed white line that separates regions I and II corresponds to the theoretical boundary of existence for the subharmonic oscillations. In region III, the oscillations, although they exist as a solution to Eqs. (1), are not visible because the oscillator switches to the ground state. (b) The dots represent three linecuts from (a) at detuning values indicated by the white arrows; a theory fit to these three data traces is represented by the black lines. (c)-(f) Histograms of the detected solution to Eqs. (1), are not visible because the oscillator switches to the ground state. (b) The dots represent three linecuts from (a) at detuning corresponding to the theoretical boundary of existence for the subharmonic oscillations. In region III, the oscillations, although they exist as a solution to Eqs. (1), are not visible because the oscillator switches to the ground state. (d) The ground state (in the middle) and the three excited states with equal amplitudes and with phases differing by 2$\pi$/3; and (e) and (f) the three excited states. All histograms are sampled at a rate of $f_s = 10$ kHz that are presented in Fig. 5 and were measured with parameters indicated by the white squares in (a).

The quasiclassical field amplitudes,

$$a_1 = r_1 e^{i\phi_1}, \quad a_2 = \frac{r_2}{\beta} e^{i\phi_2}, \quad \theta = 3\phi_1 - \phi_2,$$

with $\beta = \sqrt{\alpha_2/\alpha_1}$, the stable solution has the form

$$r_1^2 = \frac{|\delta_1|}{\alpha_1} - \frac{3r_2^2}{2} + \frac{r_2^2|\delta_1| - 7r_2^2}{4} - \frac{\Gamma_1^2}{\alpha_1^2},$$

$$\sin \theta = \frac{\Gamma_1}{\alpha_1 r_2 r_1}, \quad \theta \in (\pi/2, \pi) \bmod(2\pi).$$

The solution (4) exists within an interval of the effective pump intensity $r_{2,\text{ext}}^2$,

$$r_2^2 \in (r_{2,-}^2, r_{2,+}^2), \quad r_{2,\pm}^2 = \frac{2|\delta_1|}{7\alpha_1} \left[1 \mp \sqrt{1 - \frac{7\Gamma_1^2}{\delta_1^2}}\right],$$

and at negative red detuning from the fundamental resonator mode,

$$\delta_1 \leq -\sqrt{7}\Gamma_1.$$  

The solution (4) is finite at the boundaries of existence (6); that is, the subharmonic oscillations emerge abruptly when the boundaries are crossed. The oscillations achieve a maximum intensity,

$$r_{2,\text{max}}^2 = \frac{4}{7\alpha_1} \left(|\delta_1| + \sqrt{|\delta_1|^2 - 7\Gamma_1^2}\right).$$

That grows linearly with the detuning far from the threshold, $|\delta_1| \gg \Gamma_1$. The maximum is achieved in this region at $r_2^2 = |\delta_1|/14\alpha_1$

The effective pump strength $r_2$ is defined by a nonlinear response to the external drive $B_2$, Eq. (A8). The response exhibits instability at a weak drive, $|B_2|^2 \lesssim \beta^2 |\delta_1|^2/(18\Gamma_2 \alpha_1)$, as shown in Figs. 8 and 9, but has a regular monostable behavior at larger drive, up to the maximum value given by Eq. (A18),

$$|B_{2,\text{ext}}|^2 \approx \frac{(3\omega_1 - \omega_2)^2 |\delta_1|}{7\beta^2 \Gamma_2 \alpha_1}, \quad \delta_1 \gg \Gamma_1, \Phi < 0.4\Phi_0.$$  

The phase $\phi_2$ is defined by the phase of the drive $\phi_B$, and for the stable branch and large detuning, $\delta_1 \gg \Gamma_2$, it is approximately $\pi$ shifted from the latter (see the Appendix). This situation persists within a wide interval of magnetic flux bias, $0 < \Phi < 0.4\Phi_0$, as long as the anharmonicity of the resonator spectrum exceeds the detuning, $3\omega_1 - \omega_2 \gg \delta_1$ (see the inset in Fig. 3).
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Q [arb. units]

FIG. 5. Histograms sampled with a rate $f_s = 10$ kHz, ten times slower than in Figs. 4(c)–4(f). Here, the lines between the steady states are clearly seen. (a) The histogram measured well inside region II reveals a triangle configuration of transitions between the excited-state features. (b) The histogram measured at the border between regions II and III exhibits a “star” configuration of transitions between the ground and excited states. These histograms are measured at $\delta_1/2\pi = 1.5$ MHz and drive power as indicated by the white squares in Fig. 4(a).

C. Analysis

Using the outlined theoretical results, we are able to quantitatively analyze the details in Fig. 4. Figure 4(b) displays three linecuts of the subharmonic oscillation region taken at three different values of detuning. The oscillation amplitudes, represented by dots, show sharp onsets at the low-power edge of region II and smoother decays towards the high-power edge. The solid lines correspond to a theoretical fit. The power of the external drive is in linear units,

$$P_d = 3\hbar\omega |B_2|^2 10^{\text{Att}/10},$$

where $\text{Att}$ is the attenuation (in dB) between the generator and the resonator for the 3o drive signal and $|B_2|^2$ is given by Eq. (A17). The measured output power is

$$P_{\text{out}} = \hbar\omega |a_1|^2 2\Gamma_{1,\text{ext}} 10^{G/10}.$$  

(11)

Here, $G$ denotes the overall gain of the measurement signal $\omega$ between the resonator and digitizer. The relation between the amplitudes of modes 1 and 2 is given in Eq. (4). The fit is done by adjusting only one fitting parameter, $X = Q_{2,\text{ext}} 10^{\text{Att}/10}$, which is found in Eqs. (10) and (A17), where $Q_{2,\text{ext}} = \omega_0/2\Gamma_{2,\text{ext}}$. The other parameters are measured independently and are listed in Table I; the Kerr coefficient $\alpha_1/2\pi = 85$ kHz assumes the theory value, and the gain is estimated to be $G = 66 \pm 0.5$ dB. The best fit is achieved for $X = (9.97 \pm 0.03) \times 10^3$. From this we can calculate the photon population of the first resonator mode $|a_1|^2$. It is found that 2.4 nW of output power in Fig. 4(a) correspond to roughly 100 photons.

As seen in Fig. 4(b), the observed oscillations reach a maximum and disappear (the oscillator switches to the ground state) before they reach the theoretical maximum, Eq. (8). A comparison of the experimental and theoretical maxima reveals the scaling,

$$|a_{1,\text{max}}|^2 = 0.7|a_{1,\text{max}}|^2.$$  

Using Eq. (A18) and fitting parameters extracted from Fig. 4(b), we evaluate the boundaries of existence and the stability of the subharmonic oscillations. The upper boundary is presented by the dashed white line in Fig. 4(a). Above the dashed line, in region I, the oscillations do not exist; below this line the theory predicts the existence of oscillations and stability within all of regions II and III [the oscillation lower boundary, Eq. (6), lies far below the edge of the panel]. However, the oscillations are visible only in the narrow region II but not in region III. This can be explained by a competition between the excited states and the stable ground state. At the boundary between regions II and III, the system explores all four available states, as indicated by Fig. 4(d), and in region III the system preferentially stays in the ground state [see Fig. 4(c)]. Quantitative evaluation of the lower boundary of visibility of the subharmonic oscillations requires a dynamical analysis including the effect of noise, which goes beyond the scope of the present study.

In Fig. 4 all data are taken at zero magnetic flux, $\Phi = 0$. However, the subharmonic oscillations are detected also at nonzero flux up to $\Phi \approx 0.4\Phi_0$. In Fig. 6(a) we present the maximum output power as a function of detuning for different flux bias values. At $\Phi = 0$ this corresponds to the white solid line in Fig. 4(a). The output power is proportional to the maximum population of the first resonator mode and grows linearly with the detuning, in good agreement with the theory [Eq. (8)]. Furthermore, the flux dependence of the line slopes in Fig. 6(a) is in good agreement with the theory prediction given by the flux dependence of the Kerr coefficient (A6) in Eq. (8) and making use of the scaling, Eq. (12), as illustrated in Fig. 6(b).

The subharmonic oscillations are predicted to start at a threshold at small red detuning, Eq. (7). Experimentally, this threshold is defined as the end point of the white curve in Fig. 4(a), marked with a white star. Experimental data for the frequency thresholds at different $\Phi$ are presented in Fig. 6(c) (blue dots). For smaller flux values, $|\Phi| \lesssim 0.4\Phi_0$, the threshold values of the output radiation, Eq. (11), evaluated for $\delta_1 = -\sqrt{3}/2\Gamma_{1,\text{ext}}$ exceed the noise level, $P_n \approx 0.44$ nW, and therefore the measurement procedure identifies the true threshold, Eq. (7). However, at the edges of this region, $|\Phi| \approx 0.4\Phi_0$, the output power rapidly decreases, as indicated in Figs. 4(a) and 4(b), and therefore the visible oscillation threshold shifts to larger detuning. Quantitatively, the shifted position of the threshold is defined by $P_{\text{out}}(|a_{1,\text{max}}|^2) = P_n$. We compute the solution to this equation using the parameters in Table I and the flux dependence of $\Gamma_{1,\text{ext}}$ in Eq. (A5). The internal losses and noise power are assumed to be flux independent. The theory plot of this solution, the red line in Fig. 6(c), excellently reproduces the data. We note that no fitting parameters were used in Figs. 6(b) and 6(c).

IV. CONCLUSION

We observed period-tripling subharmonic oscillations in a driven nonlinear multimode microwave resonator in the quantum regime. When an external drive tone was applied at a frequency $3\omega$, we observed output oscillations at $\omega$, demonstrating period tripling. The output signal consists of three correlated beams having the same amplitudes but with their phases shifted by $2\pi/3$ rad with respect to each other. The oscillations are observed at red detuning from the
resonator fundamental mode and in a finite interval of drive power. Due to the proximity of the second resonator mode to the drive tone, the down-conversion efficiency is strongly enhanced, enabling access to the subharmonic oscillation regime. A theory for the two-mode subharmonic resonance was developed to explain the observations. The theoretical predictions are in good quantitative agreement with the experimental observations regarding the boundary of existence of oscillations, maximum output power, and frequency threshold.

Our successful implementation of an intermode interaction of the $a_1^{\dagger}a_2$ type may in the future be used to create multiphoton entanglement and multicomponent macroscopic cat states [12].

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APPENDIX

In this appendix we derive quasiclassical solutions for Eq. (1), identify the stable solutions, and discuss the solution properties relevant for quantitative interpretation of the experimental data.

Before proceeding with solving Eq. (1), we reproduce the spectral equation for the tunable resonator [26,31] that is used for fitting the data in Fig. 3 and justifies the two-mode model for the resonator,

$$(k_o d) \tan k_o d = \frac{2 E_J(\Phi)}{E_{L, cav}} - \frac{2 C_J}{C_{cav}} (k_o d)^2. \quad (A1)$$

Here, $k_o = \omega_0 / v$ is the mode wave vector, $d$ is the length of the resonator, $E_{L, cav}$ is the inductive energy of the resonator, $C_{cav}$ is the resonator capacitance, $C_J \ll C_{cav}$ is the Josephson junction capacitance, and $2 E_J(\Phi) = 2 E_J \cos(\pi \Phi / \Phi_0)$ is the Josephson energy of the SQUID.

It is useful to note that the quasiclassical version of the Hamiltonian (2), a metapotential, can be written in terms of quadratures, $[p_n = \text{Re}(\alpha_n), \, q_n = \text{Im}(\alpha_n)]$, of the form

$$H(p_n, q_n)/\hbar = -\sum_n \left[ k_n (p_n^2 + q_n^2) + \omega_n (p_n^2 + q_n^2)^2 \right] - 2\alpha (p_n^2 + q_n^2) + \frac{\bar{\alpha}}{3} [q_1 q_2 (q_1^2 - 3 p_1^2) - p_1 p_2 (p_1^2 - 3 q_1^2)]. \quad (A2)$$

The phase portrait for the period-tripling subharmonic oscillator defined by this metapotential is presented in Fig. 7. It gives general information about the structure of the subharmonic oscillator stable steady states: they consist of four states, including the trivial ground state at the origin, $p_1 = q_1 = 0$, and the three nontrivial states corresponding to the excited oscillator.

To establish the stability of the trivial solution to Eq. (1), $a_1 = 0$, we linearize this equation and assume time dependence

FIG. 6. (a) Growth of the intensity of subharmonic oscillations with red detuning for five different flux bias values. (b) Slopes of the data traces in (a) plotted versus magnetic flux (blue circles); the red line is the theory prediction [Eqs. (8) and (12)]. (c) Threshold detunings for subharmonic oscillations at different magnetic flux values; blue dots show experimental data, and the red line shows theory (see comment in the text).

FIG. 7. Phase portrait for the subharmonic dynamics of the resonator fundamental mode defined by the metapotential, Eq. (A2), with fixed values $p_1$ and $q_1$ given by the experimental data point $\delta_1/2\pi = -12 \text{ MHz}$ and $|B_3|^2 = 6.25 \times 10^{10} \text{ photons/s}$. 

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FIG. 8. Response of the second mode $r_2^2$ as a function of detuning for different drive amplitudes $B_2$ (represented by the color scale). The phase of the response is included in the sign of the drive amplitude. The panels illustrate the evolution of the response with decreasing spectrum anharmonicity. The curves are restricted to the region of existence of subharmonic oscillations. Exact resonance ($B_2 = 0$) is indicated with a black dashed line. ($\gamma_1 = 1.93, 0.10, 0.08; \gamma_2 = 0.$)

of the small fluctuation, $a_1 \propto e^{i\omega t}$; then we find

$$\lambda_0 = i(\delta_1 + 2a_1|a_1|^2) - \Gamma_1.$$  \hfill (A3)

Since $\text{Re}(\lambda_0) < 0$, the trivial solution is always stable.

Solving Eq. (1) consists of two steps. First, a solution for the subharmonic oscillations of the first mode is constructed treating the field of the second mode as an effective pump [13,14]. Then the field of the second mode is computed as a nonlinear response to the drive. Analysis of Eq. (1) is convenient to perform using dimensionless parameters,

$$\delta = \frac{\delta_1}{\alpha_1}, \quad \Delta = \frac{3\omega_1 - \omega_2}{\alpha_1}, \quad \delta_2 = \frac{3\delta + \Delta}{\alpha_1}, \quad \gamma_2 = \frac{\Gamma_2}{\alpha_1}.$$  \hfill (A4)

Derivations of the explicit equations for the external damping,

$$\Gamma_{\text{ext}} = \omega_e(k_d d) \left( \frac{C_e}{C_{\text{ext}}} \right)^2,$$  \hfill (A5)

where $C_e$ is the coupling capacitance, and for the Kerr coefficients,

$$\alpha_a = \frac{\hbar\omega_n^2 E_{\text{ext}}^2}{16E^2(\Phi)},$$  \hfill (A6)

are found in [26]. With these parameters and using the representation (3), the stationary equation (1) takes the form

$$\begin{align*}
(\delta + i\gamma_1 + r_1^2 + 2r_1^2)\frac{\partial r_1}{\partial t} + r_2r_1^2 e^{-i(3\Phi_1 - \Phi_2)} = 0, \quad (A7) \\
\left[3\delta + \Delta + i\gamma_2 + \beta^2(r_2^2 + 2r_1^2) + \frac{\beta^2}{3}r_1^2 e^{i(3\Phi_1 - \Phi_2)} \right]r_2 + 2r_2 B_2 e^{-i\Phi_2} = 0.
\end{align*}$$

To solve Eq. (A7), we separate the real and imaginary parts,

$$\gamma_1 = r_2r_1 \sin(3\Phi_1 - \Phi_2),$$

$$\delta + r_1^2 + 2r_1^2 = -r_2r_1 \cos(3\Phi_1 - \Phi_2).$$  \hfill (A9)

and eliminate the oscillator phase. Then we get a closed equation for $r_1$, which has the solutions

$$r_1^2 = -\left[2r_1^2 \delta + (7/4)r_1^2 - \gamma_1^2 \right].$$  \hfill (A10)

These solutions are restricted to the region defined by Eqs. (6) and (7). Equations for the phase $\Phi_1$, extracted from Eq. (A9), read

$$\sin(3\Phi_1 - \Phi_2) = \frac{\gamma_1}{r_2r_1} > 0,$$

$$\cos(3\Phi_1 - \Phi_2) = \pm \sqrt{1 - \left(\frac{\gamma_1}{r_2r_1}\right)^2} = -\frac{\delta - 2r_1^2 - r_2^2}{r_2r_1}. $$  \hfill (A11)

The solutions have a threefold degeneracy: for every given value of the phase $\Phi_2$, there are three values of the subharmonic oscillation phase $\Phi_1$ shifted by $2\pi/3$ rad with respect to each other (see Fig. 7).

To evaluate the stability of these solutions, we use their simplified forms for brevity, which is valid away from the threshold, $|\delta| \gg |\gamma_1|,$

$$3\Phi_1 - \Phi_2 = 0, \quad \pi, \quad r_1 = \mp \frac{r_2^2}{2} \pm \sqrt{|\delta| - \frac{7}{4}r_2^2}.$$  \hfill (A12)

(The minus and plus signs in front of the parentheses correspond to the zero and $\pi$ phase differences, respectively.) The linearized equation for the small fluctuation $\delta a_1$ around each of the steady-state solutions has the form

$$i\delta a_1 + \left(\delta + 2r_1^2 + 2r_2^2\right)\delta a_1 - \left(r_1^2 + 2\delta + 4r_2^2\right)\delta a_1^* = 0.$$  \hfill (A13)

Assuming $\text{Re}(\delta a_1), \text{Im}(\delta a_1) \propto e^{i\omega t}$, for this solution we find from Eqs. (A13) and (A12)

$$\lambda_1^2 = \pm 6r_2^2 r_1 \pm r_2/2.$$  \hfill (A14)

For the minus sign in front of the parentheses, which corresponds to $3\Phi_1 - \Phi_2 = \pi$ in Eq. (A12), the exponent is

$$\lambda_1^2 = -6r_2^2 \left(\pm \sqrt{|\delta| - \frac{7}{4}r_2^2} \right).$$  \hfill (A15)

For the positive root, $\lambda_1^2 < 0$; hence the solution with both signs positive in Eq. (A12) is stable. The full form of this solution is presented in the main text in Eqs. (4) and (5). The other choices of the signs result in positive $\lambda_1 > 0$, hence corresponding to unstable solutions.

Equation (A8) describes a Duffing oscillator perturbed by the back-action of the subharmonic oscillator. The imaginary part of this equation defines the phase of the response $\Phi_2.$
Similar to Eq. (A11), the difference between this phase and the phase of the drive $\phi_B$ is defined by the damping $\gamma_2$, and for the major parameter interval of interest, $\Delta, |\delta| \gg \gamma_2$, phase $\phi_2$ is either close to the phase of the drive or shifted by $\pi$, $\phi_2 - \phi_B \approx 0, \pi$ [compare Eq. (A12)]. The amplitude of the response is found from the equation

$$
\left[ \left( 3\delta + \Delta + \beta^2(r_2^2 + 2r_1^2) \right) \gamma_2 - \frac{\beta^2}{3} r_1^3 \right] + \gamma_2^2 r_2^2 = \frac{\beta^2 \gamma_2,ext}{\alpha_1} |B_2|^2.
$$

(A16)

The dependence $r_2^2(\delta)$ for different drive amplitudes $B_2$ and flux values is illustrated in Fig. 8. For better clarity the plots are made neglecting damping of the second mode on the left-hand side and including the phase of the response in the sign of the drive amplitude; then positive $B_2 > 0$ correspond to $\phi_2 = \phi_B$, and negative $B_2 < 0$ correspond to $\phi_2 = \phi_B + \pi$. The response qualitatively resembles the one of the Duffing oscillator; the similarity is most pronounced at small values of the spectrum anharmonicity illustrated in Figs. 8(b) and 8(c) for $\Delta = 50$ and $\Delta = -10$. Here, the bistability region is seen at $B_2 > 0$ as well as the exact resonance, $B_2 = 0$, which is indicated with a black dashed line in Fig. 8(c). The stable solutions correspond to the lower branch at positive $B_2$ and the branch with negative $B_2$ above the resonance. There is, however, a second resonance that appears at smaller values of $r_2$; the states below this resonance line, at negative $B_2$, are unstable. The dependence $r_2^2(|B|^2)$ from Eq. (A8) is illustrated in Fig. 9 for a representative value of the detuning, $\delta = -20$, and for different values of the spectrum anharmonicity $\Delta$, which is controlled by the bias magnetic flux $\Phi$. When the spectrum anharmonicity is large, $\Delta = 1500, 500 (\Phi = 0.1, 0.32 \Phi_0)$, the stable solution for $r_2$ exists for all drive amplitudes except for very small values, where the second, unstable solution appears (this solution corresponds to the region below the second resonance in Fig. 8). In this region of large anharmonicity, which significantly exceeds the experimental interval of detunings, Eq. (A8) can be significantly simplified by dropping $|\delta| \ll \Delta$, $r_1^2, r_2^2 \ll |\delta|$, $\Delta^2 r_2^2 = \frac{2\beta^2 \gamma_2,ext}{|\alpha_1|} |B_2|^2$.

(A17)

Inserting Eq. (8) into this equation, we obtain the maximum drive power at which the subharmonic oscillations may persist,

$$
|B_2|^2 \approx \frac{2|\Delta|^2|\delta_{ext}|}{7\beta^2 \gamma_2,ext}, \quad |\delta| \gg \gamma_1.
$$

(A18)

When the anharmonicity decreases ($\Delta \lesssim 50, \Phi \approx 0.41$), an unstable (back-bending) branch emerges at large drive. This feature is associated with the bifurcation in Fig. 8. This effect should lead to a reduction of the visible part of the subharmonic oscillation region in Fig. 4(a). With a further decrease of the anharmonicity the subharmonic oscillations should disappear at $\Phi \gtrsim 0.4 \Phi_0$.


Period multiplication in a parametrically driven superconducting resonator
Period multiplication in a parametrically driven superconducting resonator

Ida-Maria Svensson,1, a) Andreas Bengtsson,1 Jonas Bylander,1 Vityal Shumeiko,1 and Per Delsing1, b)

Microtechnology and Nanoscience, Chalmers University of Technology, SE-41296 Göteborg, Sweden

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We report on the experimental observation of period multiplication in parametrically driven tunable superconducting resonators. We modulate the magnetic flux through a superconducting quantum interference device, attached to a quarter-wavelength resonator, with frequencies \( n \omega \) close to multiples, \( n = 2, 3, 4, 5 \), of the resonator fundamental mode and observe intense output radiation at \( \omega \). The output field manifests \( n \)-fold degeneracy with respect to the phase, the \( n \) states are phase shifted by \( 2\pi/n \) with respect to each other. Our demonstration verifies the theoretical prediction by Guo et al., and paves the way for engineering complex macroscopic quantum cat states with microwave photons.

The technology of circuit quantum electrodynamics\(^2\) offers an excellent platform for observation and exploitation of parametric oscillation phenomena in the quantum domain. By connecting Josephson elements to superconducting resonators, one is able to induce nonlinearity of the electromagnetic field and realise temporal high frequency control of the resonator parameters.\(^4\) \(5\) \(6\)

This in combination with high quality factors of the superconducting resonators makes the parametric oscillation regime, above the instability threshold, easily accessible with relatively small modulation intensities. Furthermore, low temperatures in the range of 10 mK allows to investigate the quantum properties of the oscillator states. Using this technique, both the degenerate (pumping at twice a resonator mode frequency) and non-degenerate (pumping at the sum of two resonator mode frequencies) parametric oscillations, have been experimentally investigated.\(^7\) \(8\)

An inherent property of parametric oscillations is phase degeneracy of the oscillator states. The non-degenerate oscillator has a continuous phase degeneracy, while the degenerate oscillator exhibits a discrete, two-fold degeneracy, which is manifested by two correlated \( \pi \)-shifted steady states.\(^9\) \(11\) In the quantum regime these states form coherent superpositions of optical coherent states, cat states,\(^12\) \(13\) which can be used as building blocks for a photonic quantum processor.\(^14\)

Quantum properties of the degenerate parametric oscillations motivate a great interest in finding ways to engineer more complex multiply degenerate oscillator states. The period multiplication phenomenon in nonlinear oscillators\(^15\) \(16\) offers an attractive approach to the problem. In recent papers, an experimental demonstration of the period tripling in a superconducting resonator was reported,\(^17\) and quantum properties of the emerging three-fold degenerate state were theoretically investigated.\(^18\) In that experiment, the self-sustained oscillations of the resonator mode were excited by injecting an external signal with a frequency close to three times the fundamental mode frequency; the excitation mechanism involves, first, an excitation of a higher resonator mode with frequency close to the driving frequency, and then a parametric down-conversion of this excited mode field. Another method for period multiplication was theoretically proposed by Guo et al.,\(^1\) based on a modulation of the oscillator nonlinearity rather than the frequency. If the oscillator potential contains a nonlinear term, \( \propto \epsilon q^{n+1} \), then a modulation of the coefficient \( \epsilon \), with a frequency \( n \omega \), will produce \( n \)-fold degenerate self-sustained oscillations with frequency \( \omega \).

In this Letter we report an experimental demonstration of period multiplication in a parametrically driven superconducting resonator, and the emergence of multiply degenerate, self-sustained oscillations. To do this we use a frequency-tunable, quarter-wavelength coplanar waveguide microwave resonator.\(^4\) \(6\) The resonator is capacitively coupled to a transmission line at one end, and grounded at the other end via a superconducting quantum interference device (SQUID). Due to the nonlinearity of the SQUID, the frequency spectrum of the resonator is non-equidistant, the lowest mode having a frequency close to 5 GHz. A similar device has been used earlier to demonstrate degenerate parametric oscillations by modulating the SQUID inductance at two times the frequency of the resonator mode. Note that the modulation of the SQUID inductance not only affects the quadratic term, i.e., the resonator frequency, but also all the higher order nonlinear terms of the Taylor expansion of the differential inductance,

\[
L_{SQ}(\phi) = \frac{L_{SQ,0}}{\cos(f/2) \cos \phi - (E_-/E_+) \sin(f/2) \sin \phi}.
\]

Here \( f(t) = 2\pi \Phi(t)/\Phi_0 \) is a normalized applied magnetic flux, \( \Phi_0 = h/2e \) is the flux quantum, \( E_\pm = (E_{j1} \pm E_{j2})/2 \) are combinations of Josephson energies of the two SQUID junctions, and \( L_{SQ,0} = (h/2e)^2(1/2E_+) \) is the SQUID inductance at zero flux. Thus, by choosing an appropriate modulation frequency, one can selectively address any of the nonlinear terms and hence implement the period multiplication proposed in Ref. [1].

The samples used are fabricated using standard microfabrication techniques. The circuits are etched in a

\[\text{a)ida-maria.svensson@chalmers.se} \]
\[\text{b)per.delsing@chalmers.se} \]
thin niobium film on c-plane sapphire substrates. The SQUIDs are made of aluminium, and fabricated using two-angle evaporation. The measurements are performed at low temperatures, $\hbar \omega \gg k_B T$, in a dilution cryostat, where the base temperature is around 12 mK. Our measurement setup is sketched in Fig. 1. We use two input lines, one for an external probe signal and one for flux modulation. Both are attenuated to reduce thermal noise. A static magnetic field can be applied using a superconducting coil, mounted close to the sample.

The sample is measured using a reflection setup. A directional coupler is employed to route the signal, and isolators protect the resonator from amplifier noise. The output signal is amplified by a cryogenic amplifier and isolated. A static magnetic field can be applied using a superconducting coil, mounted close to the sample.

The chip is mounted in a sample box at the mixing chamber stage of a dilution cryostat with a base temperature of 12 mK. Our measurement frequency is placed close to the fundamental mode frequency. $\omega_T$, which is typical for parametric oscillations, as explained theoretically\cite{11,17} and observed experimentally for the degenerate parametric oscillator\cite{7} and for the period-

FIG. 1. Schematic of the measurement setup. The resonator can be excited via an external probe or via the flux pump line. The chip is mounted in a sample box at the mixing chamber stage of a dilution cryostat with a base temperature of 12 mK. Measurements are done using heterodyne detection.

FIG. 2. Subharmonic oscillations generated by applying a microwave signal to the flux pump line at multiples $n$ of the fundamental mode frequency. (a,b) $n = 2$; (c,d) $n = 3$; (e,f) $n = 4$; (g,h) $n = 5$.

of the oscillator states. The plots in the right column of Fig. 2 show spots that are symmetrically displaced from the origin, i.e. have equal intensities, $P = I^2 + Q^2$, and are phase shifted by $2\pi/n$ with respect to each other. Overall orientations of the multiplets are defined by the phase of the pump signal (as will be shown below), which we do not control in this experiment. Hence, the orientations of different histograms are random.

The histograms in the right column of Fig. 2 were measured in the well-established oscillator regime with pump amplitude well above the instability threshold. Those to the left were measured closer to the threshold, and here we also see a central spot representing the oscillator ground state. This is a multistability region, which is typical for parametric oscillations, as explained theoretically\cite{11,17} and observed experimentally for the degenerate parametric oscillator\cite{7} and for the period-

tripling oscillator. The multistability is explained by the fact that the oscillator ground state maintains stability and may coexist with the excited states. In histograms (a) and (e) the excited states are not well resolved. This is due to large critical fluctuations in the parameter region, where the coherent oscillations emerge having a small relative intensity. Transitions between the different states occur continuously. In histogram (f) these transitions are observed as faint lines between the spots. This is an averaging effect of stochastic switching occurring during sampling.

To explain our observations we analyze the multimode Hamiltonian of the SQUID \(^{11}\),

\[ H = \sum_n \hbar \omega_n \hat{a}_n \hat{a}^*_n + V(\dot{\phi}, t), \]

where \( \omega_n \) is the frequency of the \( n \)-th eigenmode, \( \hat{a}_n \) and \( \hat{a}^*_n \) are the mode annihilation and creation operators, and the sum goes over all resonator modes. The variable \( \phi(t) \) refers to a dynamic deviation from the static value, \( \phi_0 = -\left( E_- / E_+ \right) \tan(F/2) \), of the phase at the SQUID-terminated edge of the resonator, \( \phi(x = d, t) = \phi_0 + \phi(t) \). Here, \( F = 2\pi \Phi_0 / \Phi_0 \) indicates a normalized static magnetic flux, and \( d \) denotes the resonator length. An expansion of \( \phi \) over the cavity modes reads,

\[ \dot{\phi} = \sum_n \beta_n (\hat{a}_n + \hat{a}^*_n), \quad \beta_n = \gamma \sqrt{\frac{8\pi Z_0 k_n d}{R_K}}, \]

where \( Z_0 = \sqrt{L_0 / C_0}, \quad R_K = h / e^2 \), and \( k_n = \omega_n / \nu \) is the mode eigenvector defined by the spectral equation \(^{11}\), \( k_n d \tan(k_n d) = 1 / \gamma \), where \( \gamma = E_{L, \text{cav}} / (2E_+ \cos(F/2)) \ll 1 \) is the participation ratio of the SQUID inductance versus the cavity inductance. Here, \( L_0 \) and \( C_0 \) are the inductance and capacitance per unit length of the coplanar waveguide transmission line, respectively. Together they define the phase velocity in the waveguide resonator, \( \nu = 1 / \sqrt{Z_0 C_0} \). The potential energy \( V(\dot{\phi}, t) \) in Eq. (2) represents the nonlinear part of the inductive energy of the SQUID and has the form for an asymmetric SQUID,

\[ V(\dot{\phi}, t) \approx -E_+ \cos(F/2) \left( \cos \dot{\phi} + \frac{\dot{\phi}^2}{2} \right) + \delta f(t) \left[ E_+ \sin(F/2) \cos \dot{\phi} + \frac{E_- \cos(F/2) \sin \dot{\phi}}{\cos(F/2)} \right], \]

where \( \delta f(t) = 2\pi \Phi_0(\dot{t}) / \Phi_0 \) describes the pump, i.e. a small-amplitude temporal modulation of the applied magnetic flux, which we here consider in the linear approximation. Other adopted approximations in Eq. (4) include the assumption of a small SQUID asymmetry, \( E_- \ll E_+ \ll 1 \), and that the effect of the SQUID capacitance is small.

It is appropriate to make the following comments to Eq. (4): (i) the symmetric part of the potential, \( \propto E_+ \), contains only even powers of \( \phi \), which implies that only even subharmonics can be excited by the flux pump if the SQUID is perfectly symmetric. (ii) The asymmetric part, \( \propto E_- \), is responsible for the excitation of odd subharmonics; this part also contains a linear term, \( \propto \phi \), which implies that the nonstationary magnetic flux directly generates a field inside the resonator, i.e. acts as an effective current drive, in addition to the parametric pump. This field is down-converted via the mechanism discussed in detail in \(^{17}\) and contributes to excitation of odd subharmonics: for odd subharmonics, a higher resonator mode lies close to the driving frequency, and this mode is efficiently excited by the drive and acts as a pump exciting the subharmonics. (iii) There is a possibility of a parasitic crosstalk between the flux pump line and the resonator that may effectively give a current drive of the resonator at the pump frequency. Therefore, it is possible to get both odd and even subharmonics, even if the SQUID is completely symmetric. Both the crosstalk and the asymmetry of the SQUID are difficult to estimate and this makes it hard to make a quantitative comparison to the theory.

Suppose the magnetic flux is modulated with a frequency \( n \omega = n(\omega_1 + \delta) \) and \( \delta f(t) = \delta f_0 \cos nt \). Then we assume, focusing on the resonant response of the fundamental resonator mode \( n = 1 \), the field in the cavity to be a superposition of two harmonics,

\[ \phi(t) = \beta_1 \hat{a}_1(t) e^{-i\omega t} + \beta_2 \hat{a}_2(t) e^{-i\omega t} + \text{H.c.} \]

where \( \hat{a}_n \) refers to a response of a mode with a frequency close to the pump frequency. A quasicausal approximations is relevant for large-amplitude subharmonic oscillations far from the threshold. Following the method of Refs. \(^{11,17}\) we derive a shortened dynamical equation for a slowly varying quasicausal amplitude \( a_1(t) \) in the rotating wave approximation. This equation has the following universal form for all integers \( n \),

\[ i \dot{a}_1 + (\delta + i \Gamma_1 + \alpha |a_1|^2) a_1 + \epsilon_n a_n^{*-1} = 0. \]

The coefficients, \( \alpha = (E_+ / h) \cos(F/2) \beta_1^2 \) and \( \epsilon_4 = (E_- / h) \cos(F/2) \beta_3 \delta f_0 \), are familiar from the study of degenerate parametric resonance. \(^{11}\) The pump coefficient for period quadrupling, \( \epsilon_4 \), is expressed through \( \epsilon_2 \), \( \epsilon_4 = -\epsilon_2 (\beta_3^2 / 2) \). Similar scaling holds for higher even-order coefficients, \( \epsilon_{2k} \propto \epsilon_2 \beta_1^{2k-2} \). The pump coefficient for period tripling consists of two contributions,

\[ \epsilon_3 = \frac{E_-}{2 h} \beta_1^2 \delta f_0 + \frac{E_+}{h} \cos(F/2) \beta_2^2 \beta_3 a_3. \]

The first term is the direct effect of the flux pump and it only exists for an asymmetric SQUID. The second term results from the secondary, current-driving effect of the higher mode; this term exists for both symmetric and asymmetric SQUIDs. All higher odd-order coefficients have similar structure with the scaling factor \( \beta_1^{2k-2} \).

The stationary solutions of Eq. (6) for flux pumping at the \( n \)-th multiple of the fundamental resonator node have the form \( a_{1,n} = r_n e^{i\gamma_n} \), with \( \sin(n \theta_n - \arg \epsilon_n) =
The outlined calculations justify the relevance of the model\(^1\) for parametrically driven superconducting resonators. They provide explicit equations for the model coefficients for this setup and qualitatively explain our experimental observations.

The nonlinear parametrically driven oscillator has indeed a very rich behavior. Further insight into the properties of period multiplication can be gained by exploring regimes beyond the observed steady-state multiplets. In Fig. 3 we present a histogram of the period-tripling oscillations measured at larger red detuning, where the oscillation intensity is substantially stronger than the one in the histograms in Fig. 2. Here we detect coexistence of two triplet states, with different amplitudes and different orientations, presumably corresponding to a higher order nonlinearity.

Injection of a weak probe signal into the resonator is known to produce a phase-locking effect, observed for example in the degenerate parametric oscillator\(^{19}\). There, the effect was manifested by a gradual disappearance of one of the doublet states, and explained by a symmetry breakdown under on-resonance injection. Similarly, in the period tripling regime we observe a gradual disappearance of the triplet components when an on-resonance probe signal is applied, see Fig. 4(a) and (b).

Furthermore, injection of a probe signal slightly detuned, by 1 Hz, produces a completely different, dramatic effect on the period tripling oscillations, see Fig. 4(c) and (d). Instead of breaking the symmetry of the overall triplet pattern, the individual round spots deform into crescents with a size that increases with increasing probe intensity. Such a behavior resembles the phase locking effect observed in non-degenerate parametric oscillators\(^{8,10}\), where the oscillator state possesses continuous phase degeneracy. This resemblance leads us to interpret our observation as the result of a deformation of each stable stationary state of the triplet into a stable cycle with simultaneous phase locking. A further theoretical investigation is required to give quantitative explanations to these observations.

In conclusion, we observed period-multiplication phenomena in parametrically driven superconducting resonators. We observed robust output radiation at a frequency close to the fundamental resonator mode, with \(n = 2, 3, 4, 5\) evenly shifted phase components under an applied pump signal with frequencies \(n\) times the detection frequency. Our qualitative analysis of the resonator dynamics agrees with the observations, and corroborates the model proposed in [1]. Our observations put a firm ground for further exploration of quantum aspects of the period-multiplication phenomena, and the possibility of engineering complex photonic cat states with potential applications to information technologies.

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Microwave photon generation in a doubly tunable superconducting resonator
Microwave photon generation in a doubly tunable superconducting resonator


Microtechnology and Nanoscience, Chalmers University of Technology, Göteborg, Sweden
E-mail: ida-maria.svensson@chalmers.se

Abstract. We have created a doubly tunable resonator, with the intention to simulate relativistic motion of the resonator boundaries in real space. Our device is a superconducting coplanar-waveguide microwave resonator, with fundamental resonant frequency \( \omega_1/(2\pi) \sim 5 \text{ GHz} \). Both of its ends are terminated to ground via dc-SQUIDs, which serve as magnetic-flux-controlled inductances. Applying a flux to either SQUID allows the tuning of \( \omega_1/(2\pi) \) by approximately 700 MHz. Using two separate on-chip magnetic-flux lines, we modulate the SQUIDs with two tones of equal frequency, close to \( 2\omega_1 \). We observe photon generation, at \( \omega_1 \), above a certain pump amplitude threshold. By varying the relative phase of the two pumps we are able to control this threshold, in good agreement with a theoretical model. At the same time, some of our observations deviate from the theoretical predictions, which we attribute to parasitic couplings resulting in current driving of the SQUIDs.

1. Introduction

Vacuum is commonly considered to be empty space. However, in quantum theory, it contains vacuum fluctuations of the electromagnetic field. Due to these fluctuations, two perfectly conducting mirrors at rest, placed in close vicinity of each other, can exhibit radiation pressure forces, known as the Casimir effect [1]. Furthermore, if the mirrors are moved with a speed close to the speed of light, real photons can be generated as excitations of the vacuum fluctuations, a phenomenon called the dynamical Casimir effect (DCE) [2]. In fact, photon generation through the DCE requires only one rapidly moving mirror to produce photons [3, 4].

Using superconducting circuits, the physical conditions equivalent to a mirror moving at about 1/4 of the speed of light can be created [5]. This is done by placing a superconducting quantum interference device (SQUID) at the end of a transmission line. The SQUID acts as a tunable inductance, \( L_s(\Phi_{\text{ext}}, I_s) = \Phi_0/\left(2\pi|\cos(\Phi_{\text{ext}}\pi/\Phi_0)|\sqrt{T_c^2 - I_s^2}\right) \), where \( \Phi_0 \) is the magnetic flux quantum, \( \Phi_{\text{ext}} = \Phi_{dc} + \Phi_{ac}(t) \) is the applied external magnetic flux, \( I_c \) is the SQUID’s critical current, and \( I_s \) the current through the SQUID. The SQUID inductance can be modulated either by flux pumping, through \( \Phi_{ac} \), which is a direct modulation of the resonator boundary condition and the analogue of a moving mirror, or by ac driving the SQUID current \( I_s \). The generation of DCE photons using a flux-pumped SQUID at the end of a transmission line was suggested in Ref. [6] and demonstrated in Ref. [7].
If a SQUID is included in a resonator and flux-modulated around twice the resonant frequency, the system is the equivalent of a parametric oscillator (PO) [8–10], i.e., a harmonic oscillator driven by the modulation of a system parameter, here the resonant frequency. The PO has a flux-pump amplitude threshold, determined by the system damping, above which self-sustained oscillations are generated [9]. Below threshold, the system can be operated as a parametric amplifier in which small input signals near its resonant frequency are amplified [11–14].

In this paper, we use a superconducting coplanar waveguide λ/2 resonator, with each end grounded via a SQUID. If driven separately, both SQUIDs can generate photons individually through the DCE. When driven together at the same frequency, the resonator can be thought of as a vibrating resonator or a breathing resonator, depending on the phase difference between the two drive signals. When flux pumping both SQUIDs around 2ω1, theory predicts constructive interference for the breathing mode, leading to a low threshold for photon generation [9], and destructive interference for the vibrating mode, i.e., no photon generation [15–19]. In addition to investigations of the DCE, this device opens up doors for future interesting experiments, for example, measurements of the twin paradox [20], where a microwave signal could be sent on a “space trip” in a vibrating resonator, and the generation of cluster states [21].

2. Experimental setup
Our circuit is placed on a sapphire substrate (Fig. 1). The SQUIDs are made of aluminium and deposited by two-angle evaporation, while the rest of the circuit is etched in niobium. The resonator is meandered and grounded in both ends. To avoid a parasitic superconducting loop through resonator and ground plane, we made a slot in the ground plane. To keep good electrical contact we bridged the slot with normal metal (gold), see Fig. 1(b).

We use a reflection setup with circulators to allow for proper attenuation of an input probe signal and amplification of the resonator output signal (Fig. 1(d)). The flux-line setup enables both dc biasing and fast modulation (pumping) through separate lines that are combined in bias-Tees at the mixing chamber stage of the cryostat. The pump signals are generated in two sources, $P_{l/r}$, phase locked by a 10 MHz reference. To measure the phase difference between the sources, their output signals are divided in power splitters and compared using a mixer. Provided that the two pumps have the same frequency, the mixer output is a dc voltage with varying amplitude, depending on the phase difference between the pumps. The resonator output is down-converted and sampled in a digitizer, which records both the in- and out-of-phase quadratures.

3. Measurement results - Resonator characterization
We tune the resonant frequency by controlling the two dc-fluxes, $\Phi_{dc,l/r}$. The first resonator mode is probed by measuring the reflection coefficient of a microwave signal incident on the resonator.
The resonator is described by the equation \( \omega_1/2\pi = 5.160 \text{ GHz} \), \( Q_{\text{ext}} = 15.4 \cdot 10^3 \) and \( Q_{\text{int}} = 36.9 \cdot 10^3 \). (b) dc tuning of the resonant frequency by both magnetic flux biases. (c) Linelcut from (b) indicated by the black dashed line. The blue dots are data and the red line is a fit to the model in Eq. (1).

Figure 2. (a) Reflection measurement (blue dots) at \( \Phi_{\text{dc}} = (0.3, 0.3) \Phi_0 \) and a fit to the model \( S_{11} = (1/Q_{\text{ext}} - 1/Q_{\text{int}} - 2i(\omega - \omega_1)/\omega_1)/(1/Q_{\text{ext}} + 1/Q_{\text{int}} + 2i(\omega - \omega_1)/\omega_1) \). For this bias point we can extract \( \omega_1/2\pi = 5.159 \text{ GHz} \) and \( \omega_2/2\pi = 5.161 \text{ GHz} \). Importantly, this means that our pump tone, applied at \( -0.5 \), hits the difference frequency \( \omega_2 - \omega_1 \), photons are up-converted from \( \omega_1 \) to \( \omega_2 \), resulting in an observed level-avoided crossing in the reflected signal, from which \( \omega_2 \) can be determined. We list three measured points in Table 1. We conclude that the anharmonicity is much larger than the linewidth (2\( \Gamma \)) of the resonator. Importantly, this means that our pump tone, applied at frequency \( \omega_1 \), should not excite the second harmonic.

(Fig. 2(a)); the extracted resonant frequencies are presented in Fig. 2(b). The pattern is slightly tilted due to a small inductive crosstalk.

The second resonator mode is outside the frequency band of our setup, but its resonant frequency can be found using parametric up-conversion [22, 23], a two-photon process which we implement by letting a weak current-probe signal resonantly excite the first mode while simultaneously flux-pumping one of the SQUIDs at a lower frequency. When this pump tone is applied at \( \Phi_{\text{dc}} = (0.3, 0.3) \Phi_0 \), the extracted resonant frequencies are presented in Fig. 2(b). The pattern is slightly tilted due to a small inductive crosstalk.

Table 1. Two-tone spectroscopy of the second resonator mode using parametric up-conversion.

<table>
<thead>
<tr>
<th>( \Phi_{\text{dc},l}/\Phi_0 )</th>
<th>( \omega_1/2\pi )</th>
<th>( \omega_2/2\pi )</th>
<th>( (2\omega_1 - 2\omega_2)/2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.01,0.01)</td>
<td>5.459</td>
<td>10.867</td>
<td>47</td>
</tr>
<tr>
<td>(0.21,-0.19)</td>
<td>5.360</td>
<td>10.668</td>
<td>52</td>
</tr>
<tr>
<td>(0.31,-0.29)</td>
<td>5.184</td>
<td>10.323</td>
<td>45</td>
</tr>
</tbody>
</table>

The second resonator mode in Table 1, can be fitted using Eq. (1). A linelcut of Fig. 2(b) with a fit is found in Eq. (1).

\[
\omega_n d \tan\left(\frac{\omega_n d}{v}\right) \left[ 1 - \left(\frac{v}{\omega_n d}\right)^2 \left(\frac{1}{\gamma_l} - c \left(\frac{\omega_n}{v}\right)^2\right) \left(\frac{1}{\gamma_r} - c \left(\frac{\omega_n}{v}\right)^2\right)\right] = \frac{1}{\gamma_l} + \frac{1}{\gamma_r} - 2c \left(\frac{\omega_n}{v}\right)^2.
\]

The subscripts \( l/r \) correspond to the left and right SQUID, \( \omega_n \) is the frequency of mode \( n \), \( d = 10.133 \text{ mm} \) is the resonator length and \( v = 1/\sqrt{C_0 L_0} \) is the phase velocity. \( \gamma_{l/r} = L_{\text{ij}/(L_0d)} \) is the inductive participation ratio, where the SQUID inductance is \( L_{\text{ij/r}} = \Phi_0/(2\pi I_s \cos(\Phi_{\text{dc},l/r}/\pi/\Phi_0)) \), assuming low signal levels, \( I_s \ll I_c \), and \( \Phi_{\text{dc,l/r}} = \Phi_0/\Phi_0 \). Here we have assumed that the two SQUIDs are nominally identical, \( \gamma_0 = \gamma_0,l = \gamma_0,r \) and \( C_{l/r} = C_{\text{ij}} = C_{\text{ij,r}} \).

The two-dimensional dc tuning, Fig. 2(b), together with the measurements of the second mode in Table 1, can be fitted using Eq. (1). A linelcut of Fig. 2(b) with a fit is found in Eq. (1).
Table 2. Extracted parameters for the resonator. The inductive participation ratio is \( \gamma_0 = L_{dc,l}/L_0, \) the SQUID critical current, \( C_J \) the SQUID capacitance, and \( \xi_{dc,l} \) represents the dc-crosstalk. \( C_0 \) and \( L_0 \) are the capacitance and inductance per unit length of the coplanar waveguide. \( \omega_1 \) is the resonant frequency of the lowest mode, \( \Gamma \) is the photon loss rate and \( Q_{\text{int}} \) and \( Q_{\text{ext}} \) are the quality factors of the resonator at \( \Phi_{dc} = (0,0) \Phi_0 \). The translation between the loss rate \( \Gamma \) and the Q-values is \( 2\Gamma = \omega_1/Q_{\text{int}} + \omega_1/Q_{\text{ext}} . \)

<table>
<thead>
<tr>
<th>( \gamma_0 )</th>
<th>( I_c )</th>
<th>( C_J )</th>
<th>( \xi_t )</th>
<th>( \xi_r )</th>
<th>( C_0 )</th>
<th>( L_0 )</th>
<th>( \omega_1(0)/(2\pi) )</th>
<th>( 2\Gamma(0)/(2\pi) )</th>
<th>( Q_{\text{int}}(0) )</th>
<th>( Q_{\text{ext}}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.64</td>
<td>1.64</td>
<td>89</td>
<td>3.64</td>
<td>4.19</td>
<td>0.159</td>
<td>0.427</td>
<td>5.459</td>
<td>0.56</td>
<td>400</td>
<td>9.6</td>
</tr>
</tbody>
</table>

Fig. 3. (a) Photon down-conversion, measured with a single pump applied to the left flux line at the bias point \((0.3,0.3) \Phi_0 \). (b) Histogram taken at the point marked with a black circle in (a). We measure two \( \pi \)-shifted states. (c) Double-pump measurement, where the phase difference \( \varphi \) between the pump signals is varied. Here the SQUID bias is \((0.2,0.2) \Phi_0 \) and \( \delta = -1 \text{ MHz} \).

4. Measurement results - Pumping

By applying a pump tone to one of the ac flux lines at a frequency close to \( 2\omega_1 \), we expect parametric oscillations at \( \omega_1 \). We measure the quadrature components of the output signal, and calculate the total output power, \( P_{\text{out}} = \langle I^2 \rangle + \langle Q^2 \rangle \). Fig. 3(a) shows photon down-conversion from \( 2\omega_1 \) to \( \omega_1 \) in a range of detuning and pump power. The detuning is denoted \( \delta = \omega_p/2 - \omega_1 \), where \( \omega_p \) is the pump frequency. Furthermore, we sample the individual quadratures, \( \langle I(t) \rangle \) and \( \langle Q(t) \rangle \), and histogram \( 1 \cdot 10^5 \) samples, see Fig. 3(b). The histogram shows two stable \( \pi \)-shifted states with the same amplitude, characteristic of parametric oscillations [8, 9, 24].

We can also apply pump signals to both flux lines simultaneously. The amplitudes are adjusted such that the effective pump strengths of the two individual SQUIDs are equal. This was done by measuring single-pump thresholds, which for the bias point \((0.2,0.2) \Phi_0 \) should be equal. We find that, depending on the phase difference \( \varphi = \varphi_r - \varphi_l \), the threshold for photon generation changes, see Fig. 3(c).

The theoretical parametric oscillation threshold is \( \epsilon_{\text{th}} = \sqrt{\Gamma^2 + \delta^2} \) both for the single and double pump cases. The measured oscillation regions are asymmetric in \( \delta \), due to a pump-induced frequency shift because of the resonator nonlinearity, shifting the resonant frequency...
### Figure 4
Measurement results at SQUID bias $(0, 0.2) \Phi_0$, in both cases using a single pump, coupled closest to the left (a), and the right (b) flux line respectively. (c) Double-pump measurement of a $\lambda/2$-resonator with only one SQUID, biased at $0.18 \Phi_0$. Here $\delta = -6 \text{ MHz}$.

Towards red detuning. The threshold is reached at an effective pump strength $\epsilon_{\text{eff}} = \epsilon_{\text{th}}$. Following the formalism [10] and extending the results to the double pump case, the effective pump strength is a superposition of complex amplitudes of flux modulation in the left and right SQUIDs, $\Phi_{\text{ac},\text{l/r}} = |\Phi_{\text{ac},\text{l/r}}|e^{i\phi_{\text{l/r}}}$, so that $\epsilon_{\text{eff}} = A(\omega)(k_l \Phi_{\text{ac},\text{l}} + k_r \Phi_{\text{ac},\text{r}})$. The coefficients are, $k_{\text{l/r}} = |\tan(\Phi_{\text{dc},\text{l/r}} \pi/\Phi_0)|/\gamma_{\text{l/r}}$. This gives an expected minimum threshold and therefore maximum photon generation in the breathing mode, $\phi = 0^\circ$, but cancellation and consequently no photons in the vibrating mode, $\phi = \pm 180^\circ$, in agreement with the measured data.

### 5. Discussion
We find qualitative agreement between Fig. 3 and the doubly flux-pumped resonator theory as well as some interesting deviations. In Fig. 4(a) and (b), we present regions of photon down-conversion, at the bias point $(0, 0.2) \Phi_0$. A single pump tone is applied to the left flux line in (a) and to the right in (b). Since for (a) the pumping is around zero flux and in (b) around $0.2 \Phi_0$, different results are expected. However, the shapes of the oscillation regions in the two graphs are rather similar, although the thresholds differ by around 5 dB. Setup attenuation differences cannot explain this large number. The observation of parametric oscillations at zero flux bias is surprising, since this contradicts theoretical predictions [10]. We attribute this effect to a possible strong inductive ac crosstalk or a parasitic coupling. Even though the crosstalk at dc is negligible, it could be large at microwave frequencies, due to differences in signal distribution on the chip for dc and microwave signals. In Fig. 4(a), the pump power applied to the left pump line would actually also pump the right SQUID, with a pump power leakage of 5 dB. Assuming that the ac flux is proportional to the pump amplitude, i.e. $\Phi_{\text{ac}} \propto 10^{(P_{\text{pump}})/20}$, this corresponds to 56% crosstalk.

A parasitic coupling from the flux pump to the SQUID current could occur, due to the presence of the low impedance loop through the resonator center conductor and the ground plane. This loop is $\sim 4000$ times larger than the SQUID loop, which corresponds to a significantly larger inductance. A coupling to this loop could cause circulating currents, and thereby directly drive the SQUID current. A possible solution, making the loop less parasitic, would be to increase its impedance by changing the geometry of the gold-bridge and slot.

Another issue is the threshold pump strength. In experiments with a $\lambda/4$-resonator with identical SQUID flux-line design and similar resonant frequency, the single pump threshold is at least 20 dB higher than what is measured here. The length difference of a $\lambda/2$ and $\lambda/4$-resonator could account for maximum a few dB of difference. Therefore the differing thresholds have to be explained, either by differing pumping mechanisms or significantly differing flux line to SQUID coupling. However, the latter can be ruled out since the couplings are designed to be identical.
To find an explanation of the mentioned discrepancies, we performed a control experiment to probe the ac crosstalk. A similar resonator was fabricated with only one SQUID, i.e., the other end was shorted to ground. Both resonator ends were equipped with on-chip flux lines, to allow for double-pump experiments. Surprisingly, we observe the same qualitative behaviour, independently of whether the resonator has two (Fig. 3(c)) or one (Fig. 4(c)) SQUID. There are some differences in output power and oscillation region widths, but this is because the measurements were performed with different samples, and at different bias points and detunings. This suggests an additional mechanism of down-conversion, possibly related to the microwave field filling the cavity and producing a current-pumping effect [25] such as that used in many parametric amplifiers [11, 26]. The difference between flux and current pumping has been discussed in Ref. [24]. The phase dependence of the threshold in Fig. 4(c), could, for instance, be explained by direct interference of the two pump signals.

6. Conclusion
Using a $\lambda/2$ resonator with two magnetic-flux-tunable boundary conditions, we demonstrated photon generation by degenerate downconversion of a pump tone. When pumping with two signals at the same frequency, we observed a pump-phase dependence of the instability threshold for photon generation. This is in agreement with a theoretical model for modulation of the boundary conditions. We also observed non-ideal results attributable to ac crosstalk and parasitic couplings resulting in current driving of the SQUIDs.

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References
Twin paradox with macroscopic clocks in superconducting circuits
Twin paradox with macroscopic clocks in superconducting circuits

Joel Lindkvist,1 Carlos Sabin,2 Ivette Fuentes,2 Andrzej Dragan,3 Ida-Maria Svensson,1 Per Delsing,1 and Göran Johansson1

1Microtechnology and Nanoscience, MC2, Chalmers University of Technology, S-41296 Göteborg, Sweden
2School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom
3Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00-049 Warsaw, Poland

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We propose an implementation of a twin-paradox scenario in superconducting circuits, with velocities as large as a few percent of the speed of light. Ultrafast modulation of the boundary conditions for the electromagnetic field in a microwave cavity simulates a clock moving at relativistic speeds. Since our cavity has a finite length, the setup allows us to investigate the role of clock size as well as interesting quantum effects on time dilation. In particular, our theoretical results show that the time dilation increases for larger cavity lengths and is shifted due to quantum particle creation.

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I. INTRODUCTION

Einstein’s theory of relativity [1,2] leads to the twin paradox, in which a twin traveling at high speeds in a spaceship ages more slowly than her sibling, who stays at rest. Although constant motion is relative, the paradox is resolved by considering the acceleration experienced by the moving twin, breaking the symmetry. The fact that moving clocks tick slower is called time dilation, and it has been tested experimentally to high accuracy by observing decay rates of particles moving at relativistic speeds through the atmosphere [3] or in an accelerator storage ring [4]. Another approach for verification is based on state-of-the-art clocks, where more modest speeds are enough to create measurable time differences. Such experiments include sending atomic clocks with commercial jets on east- and west-bound paths around the world [5] and, very recently, in a ground-based laboratory where the speed of the moving-ion clock was only 10 m/s [6].

Cutting-edge experiments in circuit quantum electrodynamics (cQED) [7,8], where quantum optical effects are investigated in the interaction of artificial atoms with one-dimensional electromagnetic fields, have now reached experimental regimes beyond standard matter-radiation interactions. In particular, it has recently been suggested that it should be possible to observe relativistic quantum effects [9–11] by ultrafast modulation of the boundary conditions for the electromagnetic field. This enabled the experimental observation of the dynamical Casimir effect [12]—a long-sought theoretical prediction of quantum field theory—opening an avenue to explore relativistic effects in quantum technologies [13]. Experiments in the overlap of quantum theory and relativity are of great relevance since we lack understanding about how the theories can be unified.

In this paper, we propose a laboratory-based experiment in which the twin paradox can be simulated with velocities approaching 2% of the speed of light. By ultrafast modulation of the electric length of a superconducting cavity, the electromagnetic field inside the cavity experiences similar boundary conditions as in a cavity moving at relativistic speeds [13]. Initiating the field inside the cavity in a coherent state, the phase of this state can be used as the pointer of a clock. We show that for state-of-the-art experimental parameters, the phase shift between the twin cavities can be as large as 130 deg, which is clearly in the measurable regime.

Unlike previous setups, our scheme addresses the effects of time dilation in relativistic quantum fields. While previous studies assumed the clock to be pointlike, in our approach the clock has a length of more than 1 cm, leading to a measurably different time dilation. In that sense, this is a proposal to test the twin paradox with macroscopic quantum systems. This is interesting since a pointlike clock is only affected by the instantaneous velocity and therefore can only be affected indirectly by acceleration. However, acceleration directly affects a quantum field contained in a cavity. The acceleration of the cavity’s boundary conditions gives rise to the dynamical Casimir effect [12,14], a genuine quantum effect where motion-induced particle creation and mode-mixing among the field modes are predicted to be observable [15]. This enables us to address further questions in the overlap of quantum theory and relativity such as the study of quantum effects on finite-size relativistic clocks. Indeed our theoretical analysis shows that the dynamical Casimir effect and the spatial extension affect the rate of the clock, i.e., time dilation. We find that time dilation increases with the length of the superconducting cavity. In other words, the traveling twin ages less if his clock is larger. Particle creation gives rise to a small shift in the time dilation, highly dependent on the details of the trajectory. These effects show up as corrections to the standard time dilation seen by a pointlike clock.

Using the setup we propose, the time dilation effects predicted in the twin paradox, as well as the effects of clock size, can be readily demonstrated in accessible parameter regimes. Currently, however, we will not be able to reach the regimes (involving velocities as large as 25% of the speed of light) required to demonstrate the effects of the dynamical Casimir effect on time dilation predicted in this paper. While these regimes have already been achieved in an experiment using a single mirror in harmonic motion [12], it is more challenging to mimic a cavity of constant proper length moving in those regimes. However, given the accelerated rate at which experimental advances in cQED have developed, we expect that in the near future it will be possible to confirm our predictions concerning particle creation as well.
II. THE TWIN PARADOX WITH CAVITIES

To describe the twin paradox scenario we consider two different observers, Alice and Rob, in a 3+1-dimensional Minkowski spacetime. Alice will be inertial and stay static with respect to our laboratory frame, with Minkowski coordinates \((t, x)\). Rob, on the other hand, will undergo a round trip starting and ending at rest with respect to Alice, at the same spacetime point. We study a simple example of such a trip, composed of four accelerated segments and two segments of inertial motion (see Fig. 1). During each accelerated segment, Rob moves with constant proper acceleration \(a\). In the laboratory frame this corresponds to movement along a hyperbola in the \((t, x)\) plane. We let the duration of each segment, in the laboratory coordinates, be equal to \(t_0\). During the inertial segments, Rob moves with a constant velocity that is set by \(a\) and \(t_0\) and we denote the duration of these segments by \(t_i\). Thus, Rob’s trajectory is completely described by \(a\), \(t_0\), and \(t_i\). In the laboratory frame, the duration of the trip is \(t_t = 4t_0 + 2t_i\).

In order to compare their elapsed proper times, Alice and Rob need to carry some form of clocks. For this, we will use cavities containing quantized one-dimensional electromagnetic fields. The cavities are of constant proper length, i.e., length measured by a comoving observer. The idea is to prepare the cavities in identical coherent states. After the trip, the phase shifts in the two cavities are determined and these are used as a measure of the elapsed proper times.

In its rest frame, the cavity is constructed by inserting two perfect mirrors separated by a distance \(L\). We imagine Alice and Rob sitting at the center of their respective cavities, each of proper length \(L\). When Rob moves with constant velocity, his cavity is shorter in the laboratory frame due to length contraction. Thus, during the accelerated segments, the two mirrors must move with different proper accelerations in order for the proper length of the cavity to stay constant. More precisely, they need to move along different hyperbolas in the \((t, x)\) plane. One of the mirrors moves with greater acceleration than Rob but for a shorter time, and vice versa for the other mirror (see Fig. 1).

For an inertial observer, a one-dimensional (1D) electromagnetic field \(\phi\) defined on a Minkowski spacetime background obeys the wave equation
\[
(\partial_t^2 - c^2 \partial_x^2)\phi = 0,
\]
where \(c\) is the speed of light. The two cavity mirrors introduce Dirichlet boundary conditions \(\phi = 0\) at the points \(x = x_i\) and \(x = x_j\), with \(L = x_j - x_i\). Quantizing the field in Minkowski coordinates, we obtain a discrete set of bosonic cavity modes with mode functions
\[
u_n(\eta, \xi) = \frac{1}{\sqrt{\pi a}} \sin[\Omega_n(\eta - \xi)] e^{-i\Omega_n \eta},
\]
and frequencies \(\Omega_n = \pi n c/L\), \(n = 1, 2, \ldots\).

An observer moving with constant proper acceleration \(a\) is static in the Rindler coordinates \((\eta, \xi)\), defined by
\[
x = \frac{c^2}{a} e^{a \eta/c} \cosh(a \eta/c),
\]
\[
t = \frac{c}{a} e^{a \eta/c} \sin(a \eta/c).
\]
In these coordinates, the wave equation takes the same form as in Eq. (1). The mirrors introduce Dirichlet boundary conditions at the points \(\xi = \xi_i\) and \(\xi = \xi_j\), separated by a distance \(L = \frac{c}{a} \text{arctanh}(\sqrt{2} a)\) with respect to Rindler position \(\xi\), corresponding to a proper distance \(L\). Quantizing the field in Rindler coordinates gives rise to a set of bosonic cavity modes with mode functions
\[
u_n(\eta, \xi) = \frac{1}{\sqrt{\pi n}} \sin[\Omega_n(\xi - \xi)] e^{-i\Omega_n \eta},
\]
and frequencies \(\Omega_n = \pi n c/L\), \(n = 1, 2, \ldots\).

During Rob’s trip, the state in Alice’s cavity will simply undergo free time evolution in the laboratory frame. To relate the initial and final states in Rob’s cavity, we use Bogoliubov transformation techniques [16]. Before the trip, the modes in the cavity are described by a set of annihilation and creation operators, \(a_n\) and \(a_n^\dagger\), satisfying the canonical commutation relations \([a_n, a_m^\dagger]\) = \(b_m a_n^\dagger\). The modes in the cavity after the trip are similarly described by another set of operators, \(b_n\) and \(b_n^\dagger\), satisfying similar commutation relations. These two sets are related by a Bogoliubov transformation, defined by
\[
b_m = \sum_n (A_n^* a_n - B_n a_n^\dagger),
\]
The Bogoliubov coefficients \(A_n\) and \(B_n\) are functions of the trajectory parameters \(a\), \(t_0\), and \(t_i\) and the proper length \(L\) of the cavity. We compute the coefficients analytically as power series expansions in the dimensionless parameter \(h \equiv aL/c^2\) (see the Appendix).
The first mode in each cavity is prepared in a coherent state, with vacuum in the higher modes. Free time evolution of a coherent state corresponds to a phase rotation. Since the proper length of the cavity is preserved throughout the trip, that is true also for the mode frequencies. Thus, we can relate the accumulated phase shift during the trip to an elapsed proper time by simply dividing with the frequency of the first mode.

The state in Alice’s cavity will transform only by a phase rotation. Knowing the Bogoliubov coefficients, we can in principle fully determine the final state in Rob’s cavity. We are, however, only interested in the phase shift \( \theta \) of the first mode, given by (see the Appendix)

\[
\tan \theta = \frac{-\text{Im}(A_{11} - B_{11})}{\text{Re}(A_{11} - B_{11})}.
\]

III. EXPERIMENTAL IMPLEMENTATION

As already suggested in Ref. [13], the cQED setup used to verify the dynamical Casimir effect [12] can be expanded to simulate relativistically moving 1D cavities. A superconducting coplanar waveguide supports a 1 + 1-dimensional electromagnetic field. Terminating the waveguide through a superconducting quantum interference device (SQUID) generates a Dirichlet boundary condition for the field at some effective distance from the SQUID itself. Now, by modifying the external magnetic flux through the SQUID, this effective distance can be tuned. Thus, the boundary condition becomes that of a moving mirror. Using two SQUIDs, we can construct a cavity where both mirrors can be moved along arbitrary and independent trajectories. In particular, these trajectories can be chosen so that the relativistic motion of a cavity with constant proper length is simulated (see Fig. 2).

To realize the twin paradox cavity trajectory described above in a cQED setup, there are several experimental constraints to take into account. The mirrors can effectively be displaced a few millimeters, while the length of the cavity itself is around a centimeter. Thus, Rob will be a “shaking twin” rather than the usual twin going to another solar system. For such a short trip, the relative phase shift between the cavities is very small. We can, however, repeat the same trip many times in order to accumulate a larger relative phase. The limit on the number of times this can be done is set by the lifetime of the field excitations in the cavity. The cavity can then be filled again with photons so that the measurement can be repeated an arbitrary number of times. Moreover, the plasma frequency of each SQUID must be larger than all the other frequencies involved, limiting the effective velocities and accelerations.

As an example of what can be achieved in the cQED setup, see Fig. 3. In this example we let the microwave source play the role of Alice’s cavity. Assuming state-of-the-art arbitrary waveform generators to source the fluxes through the SQUID loops, it should be possible to make \( t_0 \) as small as 1 ns while still maintaining the required waveform. In this case, the effective acceleration is limited to \( 1.7 \times 10^{15} \text{ m/s}^2 \) if the maximal allowed flux through the SQUIDs is not exceeded. For a standard cavity length of 1.1 cm, this corresponds to \( h = 1.3 \times 10^{-3} \). For the parameter values listed above, and with \( t_i = 0 \), we predict relative phase shifts of up to 130 deg, which is detectable. This scenario would correspond to an effective cavity displacement of 1.7 mm and a maximal velocity of 1.4% of the speed of light. With \( t_i = 4 \text{ ns} \) and the trajectory being repeated 500 times, the total travel time is 2 \( \mu \text{s} \). The time difference related to the relative phase shift

\[
\Delta \phi = 2 \pi \Delta t + \frac{2 \pi}{\lambda} d(t).
\]

FIG. 2. Experimental setup. Top: The flux-tuned SQUIDs generate time-dependent boundary conditions for the cavity field, equivalent to Dirichlet boundary conditions at different effective positions. The external fluxes \( \Phi_0(t) \) correspond to effectively moving the boundary conditions the distances \( d_0(t) \). Bottom: Sketch of the circuit setup. The signal from a coherent microwave source is used to represent Alice’s clock and to fill Rob’s cavity with photons. When Rob’s cavity has been filled a set of travels is performed by flux tuning the two SQUIDs, using the external magnetic fluxes \( \Phi_0(t) \) and \( \Phi_1(t) \). After the trips, the field in the cavity leaks out and is down-converted by a mixer using Alice’s clock. A phase difference between the two clocks is then detected as a dc change at the output of the mixer.

FIG. 3. (Color online) Time dilation. Relative phase shift between Rob’s and Alice’s cavities in an experimentally feasible regime. The parameter values used are \( t_o = 1 \text{ ns}, t_i = 0 \), and \( L = 1.1 \text{ cm} \), leading to an effective cavity displacement of 1.7 mm and a maximal velocity of 0.014c. With \( t_i = 4 \text{ ns} \) and the trajectory being repeated 500 times, the total travel time is 2 \( \mu \text{s} \). Left inset: Difference between the time dilation shown by the cavity clock and a pointlike clock as a function of \( L \), normalized to the total time dilation between Alice and Rob. The blue (red) [dark gray (light gray)] curve is excluding (including) the effects of mode mixing and particle creation. Right inset: Difference in time dilation between the cases with and without particle creation, again normalized to the total time dilation. The parameter values used in the inset plots are \( t_o = 1 \text{ ns}, t_i = 0 \), and \( a = 1.7 \times 10^{15} \text{ m/s}^2 \).

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agrees with what we would obtain if Alice and Rob were instead carrying pointlike ideal clocks. Thus, we can conclude that it is challenging but possible to simulate the twin paradox scenario in a cQED setup.

IV. COMPARISON TO POINTLIKE CLOCK

Our cavity clock agrees very well with a pointlike ideal clock in the parameter regime considered above. The reason for this is that we can choose a small $h$ value and still accumulate a phase shift large enough to observe. To second order in $h$, however, we start to see a discrepancy between the cavity clock and a pointlike one. This difference is due to both the finite extension of the cavity and the fact that nonuniform acceleration leads to mode mixing and particle creation, eventually resulting in a different phase shift for the first mode. First, neglecting the latter effects, we note that a cavity clock differs from a pointlike one during acceleration only. During an accelerated segment, the proper time elapsed according to the cavity clock is

$$\tau_{\text{cav}} = \frac{\theta_a}{\omega_0} = \frac{L}{c} \arcsin\left(\frac{a t_a}{c}\right) / 2 \arctanh(h/2),$$  \hspace{1cm} (8)

while the corresponding expression for the pointlike clock, obtained by integrating Rob’s proper time over the trajectory, is

$$\tau_{\text{point}} = \frac{c}{a} \arcsin\left(\frac{a t_a}{c}\right).$$  \hspace{1cm} (9)

Thus, the ratio of the proper times is given by

$$\frac{\tau_{\text{cav}}}{\tau_{\text{point}}} = \frac{\left(h/2\right)}{\arctanh(h/2)} = 1 - \frac{h^2}{12} + O(h^4),$$  \hspace{1cm} (10)

which is smaller than one and decreases with $h$. This means that an extended clock is slowed down during acceleration. As the clock is larger, its rate becomes slower. The effects of mode mixing and particle creation depend only on changes in acceleration and are encoded in the second-order terms of the Bogoliubov coefficients (A6) and (A7).

In order to observe the higher order effects, we need to use larger $h$ values. In earlier cQED experiments [12], effective accelerations up to $4 \times 10^{17}$ m/s$^2$ have been achieved. With such accelerations, though, the time $t_a$ would have to be very short, making the effective motion of the mirrors difficult to control. What we can do instead is to increase $h$ by using larger cavities. The inset plots of Fig. 3 show the shift in time dilation due to the different effects, as a function of $L$. In order to observe the effects of clock size, we can choose $L = 6$ cm, which is easily realizable in the cQED setup. In this regime, the clock size is clearly the dominant effect and would contribute with an additional phase shift of 3 deg, which is possible to resolve in the measurement stage. For even larger $L$, the other effects start to become important, with mode mixing being the dominant one. This can be clearly seen in the right inset, where we plot the difference in phase shift between the cases with and without particle creation.

V. CONCLUSIONS

In conclusion, we have shown that using state-of-the-art superconducting circuit technology, the twin paradox can be demonstrated in a ground-based experiment at velocities approaching 1.4% of the speed of light. Using the phase of a coherent state inside the cavity as a clock pointer, we find that the time dilation produces a phase shift of up to 130 deg, which is clearly in the measurable regime. We also note that at high accelerations the extension of the clock becomes relevant: Time dilation increases with the clock’s spatial dimension. This opens up an avenue for the experimental exploration of the differences between a pointlike [3,4,6] and a physically extended clock. In the near future, we foresee that other quantum effects on clock accuracy and time dilation can be explored using squeezed cavity states. By analyzing the twin paradox in a framework of quantum field theory with boundary conditions corresponding to relativistic motion, we are able to study theoretically the interplay of quantum effects, such as the dynamical Casimir effect, in a paradigmatic relativistic effect such as time dilation. In this way we take a step further in our knowledge on the overlap between quantum theory and relativity.

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APPENDIX: BOGOLIUBOV COEFFICIENTS

To determine the Bogoliubov coefficients $A_{mn}$ and $B_{mn}$ in Eq. (6) we use techniques developed in Ref. [16]. The Bogoliubov coefficients relating the modes of an inertial observer to those of a uniformly accelerating observer are expressed as Klein-Gordon inner products [17], $\alpha_{mn} = (v_u,u_m)$ and $\beta_{mn} = -(v_u,u_n)$, where $u_a$ and $v_a$ are given by Eqs. (2) and (5). $\alpha_{mn}$ and $\beta_{mn}$ account for mode mixing and particle creation, respectively. The resulting integrals cannot be evaluated in terms of elementary functions, but to second order in $h \equiv aL/c^2$ we can write the coefficients as

$$\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)}h + \alpha_{mn}^{(2)}h^2,$$  \hspace{1cm} (A1)

$$\beta_{mn} = \beta_{mn}^{(0)} + \beta_{mn}^{(1)}h + \beta_{mn}^{(2)}h^2.$$  \hspace{1cm} (A2)

with

$$\alpha_{mn}^{(0)} = 1, \quad \alpha_{mn}^{(1)} = 0, \quad \alpha_{mn}^{(2)} = -\frac{\pi^2 n^2}{240},$$

$$\alpha_{mn}^{(0)} = 0, \quad \alpha_{mn}^{(1)} = \sqrt{mn} \left(\frac{(-1)^{m-n} - 1}{\pi^2 (m-n)^3}\right), \quad m \neq n,$$

$$\alpha_{mn}^{(2)} = \sqrt{mn} \left(\frac{1 - (-1)^{m-n}}{2\pi^2 (m-n)^3}\right), \quad m \neq n,$$

$$\beta_{mn}^{(0)} = 0, \quad \beta_{mn}^{(1)} = \sqrt{mn} \left(\frac{(-1)^{m-n} - 1}{\pi^2 (m+n)^3}\right), \quad m \neq n,$$

$$\beta_{mn}^{(2)} = \sqrt{mn} \left(\frac{1 - (-1)^{m-n}}{2\pi^2 (m+n)^3}\right).$$
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During each accelerated segment of the trip, the fundamental mode of the cavity acquires the phase
\[ \theta_a = \frac{\pi \arcsinh(\alpha t_a/c)}{2 \arctan(h/2)}, \]  
(A4)
while for an inertial segment the corresponding phase shift is
\[ \theta_i = \pi c t_i / (\gamma L), \]  
(A5)
\[ \gamma = \sqrt{(\alpha t_i/c)^2 + 1} \] being the Lorentz factor during the inertial motion. By composing transformations described by Eqs. (A1)–(A3) and their inverses, with appropriate Rindler and Minkowski time-evolution phase transformations in between, we can find the Bogoliubov coefficients relating the cavity modes before and after the trip. Only terms up to second order in \( h \) are kept.

Acting with the Bogoliubov transformation on the vector of first moments of the cavity state and tracing out the higher modes, the expression in Eq. (7) is obtained for the phase shift of the first mode, provided that the initial phase is zero. The explicit expressions for the relevant coefficients are

\[ A_{11} = \left(1 + 6\alpha_{11}^{(2)} h^2\right)e^{4i(\theta_a + 2\theta_i)} + h^2 \sum_{k=2}^{\infty} \left(\alpha_{11}^{(1)}\right)^2 \left(2e^{2i(k+3)\theta_a+2i\theta_i} + 2e^{2i(k+1)\theta_a+(k+1)i\theta_i} - 2e^{2i(k+1)\theta_a+(k+1)i\theta_i} - 2e^{2i(k+1)\theta_a+(k+1)i\theta_i}\right) \]
\[ + 2e^{2i(k+3)\theta_a+2i\theta_i} - 2e^{4i\theta_a+(k+1)i\theta_i} + e^{2i(k+1)\theta_a+2i\theta_i} + e^{2i(k+1)\theta_a+(k+1)i\theta_i} \]
\[ - 2e^{4i\theta_a} + e^{2i(k+1)\theta_a+2i\theta_i} + e^{-4i\theta_a} - 2e^{2i\theta_a + 2i\theta_i} + e^{-2i\theta_a - 2i\theta_i}, \]  
(A6)

\[ B_{11} = 2ih^2 \beta_{11}^{(2)} \left[\sin(4\theta_a + 2\theta_i) - \sin(2\theta_a + 2\theta_i) + \sin(2\theta_a) \right] + 2i h^2 \sum_{k=2}^{\infty} \left(\alpha_{11}^{(1)}\beta_{11}^{(1)}\right) \left[\sin((4\theta_a + 2\theta_i) k) \right. \]
\[ - \left. 2 \sin((3\theta_a + 2\theta_i) k) \cos(\theta_i) - 2 \sin((3\theta_a + 2\theta_i) k) \cos(\theta_i) + \sin((2\theta_a + 2\theta_i) k) \right] \sum_{k=2}^{\infty} \left(\alpha_{11}^{(1)}\right) \left[\sin((4\theta_a + 2\theta_i) k) \right. \]
\[ + \left. 2 \sin((2\theta_a + \theta_i) k) \cos(\theta_i) + \sin((2\theta_a + 2\theta_i) k) + 2 \sin((\theta_a + \theta_i) k) \right] \sum_{k=2}^{\infty} \left(\alpha_{11}^{(1)}\right) \left[\cos((2\theta_a + 2\theta_i) k) \right. \]
\[ + \left. 2 \sin((\theta_a k) \cos(3\theta_a + 2\theta_i)) - 2 \sin(\theta_i k) \cos(3\theta_a + 2\theta_i) \right]. \]  
(A7)

Paper E

Storage and on-demand release of microwaves using superconducting resonators with tunable coupling
Storage and on-demand release of microwaves using superconducting resonators with tunable coupling

Mathieu Pierre,a) Ida-Maria Svensson, Sankar Raman Sathyamoorthy, Göran Johansson, and Per Delsingb)

Department of Microtechnology and Nanoscience (MC2), Chalmers University of Technology, SE-412 96 Göteborg, Sweden

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We present a system which allows to tune the coupling between a superconducting resonator and a transmission line. This storage resonator is addressed through a second, coupling resonator, which is frequency-tunable and controlled by a magnetic flux applied to a superconducting quantum interference device. We experimentally demonstrate that the lifetime of the storage resonator can be tuned by more than three orders of magnitude. A field can be stored for 18 μs when the coupling resonator is tuned off resonance and it can be released in 14 ns when the coupling resonator is tuned on resonance. The device allows capture, storage, and on-demand release of microwaves at a tunable rate. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4882646]

At the interface between quantum optics and integrated electronics, superconducting circuits constitute a flexible and scalable platform for quantum information processing based on the manipulation of qubits and microwave photons.1–3 One of the main challenges is to develop basic tools required to manipulate photons at the single photon level. Such functionalities include, for instance, photon generation,4 detection, routing,5 and storage.5

A scalable architecture of a circuit quantum electrodynamics (circuit-QED) experiment can have the structure of a quantum network.7 In this scheme, propagating photons carrying bits of quantum information travel between nodes where quantum information is processed through the interaction of these photons with various quantum systems. Such quantum systems can be superconducting quantum bits, spin ensembles, quantum dots, or mechanical oscillators. Despite recent progress in achieving a strong coupling with propagating photons, it is convenient to embed the quantum system in a cavity where its coupling to the field is resonantly enhanced.1 In this scheme, the cavity must be able to exchange photons with its surrounding, for example, a microwave transmission line. However, the ability for a cavity to store photons for a long time is not compatible with its ability to release them fast, since these two time scales are linked.

In this article, we present a superconducting resonator which features a controllable coupling to a transmission line where photons can be emitted from the cavity, or, inversely, where incident photons can be fed to the cavity. The coupling of the resonator to its environment can therefore be dynamically tailored to a specific goal, either preserving the coherence of the quantum system or, on the contrary, enabling its fast measurement and control. Adding new features to a basic component of circuit-QED experiments such as a superconducting resonator gives rise to new possibilities. For example, making a resonator tunable in frequency has led to parametric amplifiers11,12 and oscillators.13,14 Using a tunable coupling for a resonator effectively tunes its external Q-value, and it is a way to control the emission and engineer the shape15,16 of photons initially stored in a cavity. This is useful to transfer a quantum state with arbitrarily high fidelity between distant cavities.17 Moreover, a tunable coupling can also be used for quantum bath engineering. It indeed enables to control the damping of a quantum system embedded in a cavity. This can be used for the fast initialization of a quantum bit or for the creation of arbitrary strongly squeezed states of the field.18

Our scheme uses a λ/2 frequency-tunable resonator8–10 intercalated between a λ/4 fixed-frequency resonator and a transmission line (Fig. 1). The fixed resonator will be referred to as the storage cavity, whereas the frequency-tunable resonator will be referred to as the coupling cavity. The coupling of the storage cavity to the transmission line depends on the transmission of microwaves through the coupling cavity. The latter behaves as a Fabry–Pérot cavity and thus its transmission is frequency-dependent. The key feature of the system is that the resonance frequency of the coupling cavity can be adjusted with respect to the resonance frequency of the storage cavity, which leads to a variable coupling between the storage cavity and the transmission line. It is the detuning between the two cavities which determines the strength of the coupling.

The tunability in frequency of the coupling cavity is obtained by introducing a superconducting quantum interference device (SQUID) at its voltage node, i.e., in its middle. The tunable inductance of the SQUID provides a variable contribution to the resonator total inductance, and thus makes its resonance frequency tunable. Moreover, its inductance can be tuned extremely fast.

The coplanar transmission line resonators are made to have a characteristic impedance of 50 Ω. Their center conductor as well as the ground planes is made of niobium. An 80 nm thick layer of niobium is sputtered on a high resistivity silicon substrate, after HF cleaning and annealing at 700 °C.

a)mathieu.pierre@lcnmi.cnrs.fr
b)per.delsing@chalmers.se
The niobium is etched through a UV5 resist mask defined with electron-beam lithography. We use reactive-ion etching of the two aluminum layers, and lift-off to fabricate the SQUID in a gap etched in the resonator center conductor. We place the SQUID in a gap of the niobium center conductor. The line on the right side of niobium. (c) Capacitor between the input transmission line and the coupling cavity. (d) Capacitor between the two resonators. (e) Aluminum SQUID in a gap of the niobium center conductor. The line on the right side is used to induce magnetic flux in the SQUID loop.

The niobium is etched through a UV5 resist mask defined with electron-beam lithography. We use reactive-ion etching with an NF3 plasma, followed by an oxygen plasma ashing and subsequent chemical resist removal. The SQUID is placed in a gap etched in the resonator center conductor. We use e-beam lithography, double-angle evaporation of aluminum (50 and 70 nm thick), and lift-off to fabricate the Josephson junctions. In order to get a good contact between the niobium and the aluminum SQUID, an argon-ion milling step is realized in situ in the deposition chamber, prior to the evaporation of the two aluminum layers.

The sample is placed at the cold stage of a dilution refrigerator, at a temperature below 25 mK, inside a magnetic shield. The magnetic flux in the SQUID loop is controlled both by a coil located on the sample box and with an on-chip current line. This line can be seen in Fig. 1(e). Whereas the coil is used for static tuning, the on-chip line has a large bandwidth, up to 12 GHz, and thus allows fast control of the coupling. The microwave signal reflected from the sample undergoes heterodyne demodulation, and both quadratures are digitized and sampled at 200 MS/s.

First, the devices are characterized with reflection measurements with a network analyzer as a function of the magnetic flux in the SQUID loop. The two coupled resonators give rise to two resonance modes, each of which lead to a Lorentzian-shaped signature in the reflection coefficient. From their linewidth, the coupling of these two modes to the transmission line can be extracted.

Two different samples were tested. Their properties can be found in Table I. We present data on sample A, which was optimized for obtaining a large tunability range. Sample B had a less favorable internal quality factor, probably due to small variations in the fabrication process. The natural frequency of the storage cavity is $f_{}\text{st}=5.186\text{GHz}$. The frequency of the coupling cavity is periodically tuned with the flux (Fig. 2(a)). It evolves from a maximum value of 5.36 GHz obtained when the flux in the SQUID loop is an integer number of flux quanta. It reaches a minimum value for half integer flux quanta in the SQUID loop, well below 4 GHz, the lower bound of our measurement band. The two cavities are on resonance at around a quarter of a flux quantum. This shows up as avoided level crossings, indicating that two modes arise from the coupling of the two cavities. Fig. 2(a) shows the extracted resonance frequencies $f_{1}$ and $f_{2}$ corresponding to these two modes. Their evolution is well reproduced with a simple analytical model

$$f_{z}=\frac{1}{2}(f_{1}+f_{2})\pm\sqrt{\frac{\Delta f}{2}},$$

(1)

where $f_{z}$ is the frequency of the coupling cavity, $\Delta f=f_{2}-f_{1}$ is the detuning, and $g=K_{0}(2f_{1}^{2}+f_{2}^{2})=18.3\text{MHz}$ is the coupling between the two resonators. The frequency of the coupling cavity is given\(^{11}\) by $f_{c}=f_{c0}(1+\gamma_{0}/\cos(\pi\Phi/\Phi_{0}))$, where $f_{c0}=5.810\text{GHz}$ is the geometrical frequency of the coupling cavity, i.e., as it would be without the SQUID. The participation ratio $\gamma_{0}=8.4\%$ is the ratio between the Josephson inductance of the SQUID (at zero flux) and the inductance of the cavity.

Fig. 2(b) shows the external quality factors extracted from the reflection measurement. They are defined as $Q_{\text{ext}}=\omega_{0}/2\Gamma_{\text{ext}}$, where $\omega_{0}$ is the angular frequency of the resonance mode and $\Gamma_{\text{ext}}$ is the coupling rate to the transmission line. Far from the resonance points, the two lines can be interpreted separately in terms of coupling of each resonator to the transmission line. The bottom line with $Q_{\text{ext}}\approx 100$ is the result of the large, capacitive coupling of the coupling cavity, which is fixed and set by the value of the capacitance $C_{\text{ext}}$. On the contrary, the upper line, corresponding to the coupling of the storage cavity, strongly depends on the detuning between the two resonators. The coupling indeed varies over several orders of magnitude. The storage cavity therefore evolves from a strongly overcoupled to a strongly undercoupled regime as its external quality factor can be either larger or smaller than its intrinsic quality factor of 600 000. Close to zero detuning, the two modes tend to have equal coupling to the transmission line, which explains why the two lines cross at these points.

---

**TABLE I. Properties of the two measured devices: frequency of the storage resonator, critical current of the SQUID, capacitance between the coupling cavity and the transmission line, coupling rate of the coupling cavity to the transmission line, coupling of the two resonators, internal quality factor of the storage cavity, minimum coupling time, and maximum storage time.**

<table>
<thead>
<tr>
<th>Sample</th>
<th>$f_{1}$ (GHz)</th>
<th>$I_{c}$ (μA)</th>
<th>$C_{\text{int}}$ (fF)</th>
<th>$C_{\text{ext}}$ (fF)</th>
<th>$\kappa$ (MHz)</th>
<th>$g$ (MHz)</th>
<th>$Q_{\text{ext}}$</th>
<th>$\tau_{s}$ (ns)</th>
<th>$\tau_{z}$ (μs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5.186</td>
<td>1.0</td>
<td>70</td>
<td>5.1</td>
<td>250</td>
<td>18.3</td>
<td>600 000</td>
<td>14</td>
<td>18.4</td>
</tr>
<tr>
<td>B</td>
<td>5.416</td>
<td>1.5</td>
<td>97.5</td>
<td>5.1</td>
<td>5</td>
<td>21.2</td>
<td>80 000</td>
<td>200</td>
<td>2.4</td>
</tr>
</tbody>
</table>
The evolution of the couplings with flux can be modeled with three free parameters, the two capacitances $C_{\text{out}}$ and $C_c$ and the SQUID inductance $L_S$. The transition rate describing the leak of the energy contained in each mode is the average between the transition rate for each cavity, with weights taking into account how the coupled modes of the system decompose in the uncoupled cavity modes. This translates into the following equations:

$$\Gamma^+_{\text{ext}} = \cos^2\left(\frac{\theta}{2}\right)\Gamma_{\text{ext},1} + \sin^2\left(\frac{\theta}{2}\right)\Gamma_{\text{ext},2}$$

$$\Gamma^-_{\text{ext}} = \sin^2\left(\frac{\theta}{2}\right)\Gamma_{\text{ext},1} + \cos^2\left(\frac{\theta}{2}\right)\Gamma_{\text{ext},2}$$

where $\theta = \arctan(2\gamma/\Delta f)$ is the mixing angle describing the eigenmodes of the system. The model relies on analytical expressions for the coupling of the two cavities. The coupling cavity has a coupling rate $\Gamma_{\text{ext},2} = \kappa/2 = \omega_2(Z_0\omega_2C_{\text{out}})^2/\pi$. The storage cavity has a residual coupling $\Gamma_{\text{ext},1} = (2/\pi)\omega_1(Z_0\omega_1C_{\text{out}})^2(Z_0\omega_1C_{\text{out}})^2/(1 + (\omega_1L_S/Z_0)^2)$. The best fit gives $C_c = 5.1$ fF, $C_{\text{out}} = 70$ fF, and $L_S = 345$ pH. At maximum detuning, the external quality factor $Q_{\text{ext}}$ of the storage cavity diverges because the SQUID not only detunes the coupling cavity but also enhances the scattering of the photons leaking out.

The static characterization of the system proves that the coupling of the storage cavity can be varied over several orders of magnitude. In particular, this cavity can be efficiently decoupled from the transmission line, for a sufficient detuning between the resonators, which enables to store a field inside. To demonstrate this, we performed time-resolved measurements, in which a field is built-up from an RF input pulse, stored, and released after a controlled delay. Figs. 3(a) and 3(b) show the experimental protocol. Starting at an intermediate detuning, corresponding to a coupling of 1/250 ns, a microwave pulse on resonance with the storage cavity is applied to the input port of the device. Once a steady-state field has been reached in the cavity, the detuning is increased to its maximum, with half a quantum of flux in the SQUID loop, and the input is turned off at the same time. The measured output signal goes to zero (Fig. 3(c)), which proves that the coupling is effectively off. After a delay, the coupling is brought back to its initial value. The stored energy is therefore emitted to the transmission line, which is seen with a fast increase of the output signal, followed by an exponential decay as the field leaks out to the transmission line at a constant rate.

This experiment shows that the coupling can be turned on and off with our device. Moreover, it provides a measurement of the intrinsic lifetime of the cavity. Indeed, the longer the delay is, the less signal is recovered. Fitting the exponential decrease of the recovered signals, we estimate a lifetime of 18.4 $\mu$s or, equivalently, a quality factor of $6 \times 10^5$. Note that this experiment has involved the storage of an average of 80 photons in the cavity. Moreover, we have checked that it can also be done at the single photon level, at the price of decreased signal to noise ratio.

To demonstrate that the coupling can be continuously adjusted to a desired value, the experimental protocol can be slightly modified. We now vary the detuning in the release step. As a result, we observe that the output signal decays exponentially at different rates, from a very slow decay when the detuning is large, to a very fast decay when the detuning is small (Fig. 3(d)). The amplitude of the output signal also varies accordingly, since the same amount of energy is released in all experiments. Fig. 3(f) shows the decay time as a function of the flux in the SQUID loop in the release step. This is a direct measurement of the cavity lifetime, provided that a factor $1/2$ is taken into account in order to obtain the energy decay time from the voltage decay time. At around half a quantum of flux, this time is large and saturates at the intrinsic lifetime of the cavity, which means that the energy is lost rather than being released to the transmission line. At around a quarter of flux quantum, when the detuning approaches zero, the output signal no longer shows an exponential decay (Fig. 3(e)), which prevents from relating the decay time to the lifetime of the cavity. When the two cavities are close to resonance, the stored microwave field oscillates between the two cavities. The release time is therefore limited by the transfer time, given by $1/\kappa = 13.7$ ns. Moreover, this oscillation is slightly underdamped, since the lifetime of the coupling cavity $1/\kappa$ is such that $\kappa/2\pi < 4\pi$. It therefore shows up as an oscillation superimposed on the output voltage decay. This is well reproduced by a model of the release mechanisms, which includes low-pass filtering of the output signal to account for the finite bandwidth (90 MHz) of the digitizer (Fig. 3(e)). Nevertheless, the experiment shows that the storage cavity

![Graphical representation](image-url)
The tunable coupling system presented here has several advantages. It is easy to implement in any circuit-QED layout, where it can simply replace the usual coupling capacitances. An extremely small coupling is naturally obtained when the coupling cavity is maximally detuned and the losses are limited by internal losses in the storage cavity. On the other hand, the maximum coupling is set by the two capacitances of the system. Couplings even stronger than shown in this article should be easy to obtain. Furthermore, with this coupling mechanism, the storage cavity is free from Josephson junctions, which cause additional dissipation in tunable resonators. Indeed, when the coupling is set to a small value in order to store a quantum field or perform coherent manipulation, no field exists in the tunable cavity. This may explain why we reached a higher cavity lifetime than previously reported for a cavity equipped with a different, inductive tunable coupling.\textsuperscript{6}

To conclude, we report on a superconducting circuit for coherent manipulation of microwave signals. Its primary function is to make the coupling between a microwave coplanar resonator and a transmission line tunable. This is interesting in circuit-QED experiments, for instance, to make single-photon on-demand sources or to shape photons. We demonstrated that the lifetime of a resonator can be tuned in a large range, from 14 ns to 18 µs. This proves that, while our system enables to achieve large couplings, it does not enhance the cavity losses, and therefore allows to store microwaves for a long time.

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Resonant and off-resonant microwave signal manipulations in coupled superconducting resonators
Resonant and off-resonant microwave signal manipulations in coupled superconducting resonators

Mathieu Pierre, Sankar Raman Sathyamoorthy, Ida-Maria Svensson, Göran Johansson, and Per Delsing

1Department of Microtechnology and Nanoscience (MC2), Chalmers University of Technology, SE-412 96 Göteborg, Sweden
2Laboratoire National des Champs Magnétiques Intenses (LNCMI), Université de Toulouse, INSA, UPS, UGA, CNRS UPR 3228, EMFL, FR-31400 Toulouse, France

We present an experimental demonstration as well as a theoretical model of an integrated circuit designed for the manipulation of a microwave field down to the single-photon level. The device is made of a superconducting resonator coupled to a transmission line via a second frequency-tunable resonator. The tunable resonator can be used as a tunable coupler between the fixed resonator and the transmission line. Moreover, the manipulation of the microwave field between the two resonators is possible. In particular, we demonstrate the swapping of the field from one resonator to the other by pulsing the frequency detuning between the two resonators. The behavior of the system, which determines how the device can be operated, is analyzed as a function of one key parameter of the system, the damping ratio of the coupled resonators. We show a good agreement between experiments and simulations, realized by solving a set of coupled differential equations.

In quantum technology the interaction between quantum states of light and various degrees of freedom of matter can be controlled in a variety of systems. Among them, macroscopic superconducting circuits cooled to milliKelvin temperatures are developing as a platform to manipulate microwave photons and artificial atoms. They are easy to engineer because they are integrated electrical circuits. This forms the field of circuit-Quantum ElectroDynamics (circuit-QED).

Using electrical circuits for building quantum systems allows for a precise design of Hamiltonian parameters within a wide range. Furthermore, some parameters can also be made tunable in situ, for instance, the resonance frequencies of resonators and the transition frequency of artificial atoms, also known as quantum bits. It is also essential for many experiments and applications to have tunable couplings, or equivalently, lifetimes or linewidths. Tunable couplings have already been demonstrated between qubits, between qubits and resonators, and between resonators.

In this work we focus on the tunable coupling between a resonator and a transmission line. This function is required in several types of applications. First, in quantum communication, it is envisioned that “flying” qubits are sent over long distances in the form of photons propagating between nodes acting as quantum memories and processors. These nodes could be implemented as microwave resonators coupled to qubits or other types of quantum systems. It has been shown that the transfer efficiency can be increased if one can adjust the couplings at both ends of the transmission chain. Adjusting the coupling between the transmission line and the terminating resonator to the temporal and spectral properties of the incoming wavepacket can result in full absorption, which can be viewed as an impedance matching condition for the resonator. Inversely, a resonator with tunable coupling can also be used to emit microwave photons contained in an arbitrary wavepacket. This has only been achieved with more complex schemes so far. Furthermore, it is becoming possible to simulate complex quantum systems, such as many-body states of condensed matter, using arrays of superconducting resonators and qubits. For this purpose, tunable couplings are essential to implement arbitrary Hamiltonians. Even more interestingly, dynamic processes can be studied if the couplings can be tuned fast enough, on the timescale of the processes under study.

Since resonators are either capacitively or inductively coupled to transmission lines, a first approach to make the coupling tunable is to use a tunable circuit element, such as a tunable inductance, for instance a superconducting quantum interference device (SQUID). To allow for more complex manipulations of the microwave signals, a second approach is based on a dual resonator architecture. A high quality factor resonator, dedicated to the storage of microwave radiation, which can be viewed as a quantum node, is connected to a transmission line via a low quality factor resonator. This low Q resonator permits the fast transfer, storage or retrieval, of the quantum information encoded in the microwave radiation. It has already been shown how parametric processes can be used for the coherent manipulation of the microwave signals, either by coupling the two resonators with a Josephson ring modulator, a flux-driven Josephson junction circuit, or with a superconducting qubit. In our work, we use a similar dual resonator architecture, but our approach for the coherent control is different. We made the low Q resonator frequency-tunable, and the resonators are simply capacitively coupled.

In a previous article, we demonstrated the storage of microwaves in a superconducting resonator by switching on and off this tunable coupler. We showed that microwaves can be released from the storage resonator through the frequency-tunable low-Q coupling resonator at a varying rate. We presented a sample that was engineered to show a high on/off coupling ratio. The goal of the current article is to extend this work by presenting a generic model for this coupled-resonator circuit,
valid in a large range of parameters and supported by experimental data in good agreement with the theory. We show that the behavior of each sample is governed by a single parameter, a ratio between coupling rates, which corresponds to the damping ratio of the coupled resonator system. We present an experimental comparison of two samples operating in two distinct regimes. One of the sample corresponds to the results already presented in our previous work. It is optimized for direct addressing of the storage resonator, which is done in the off-resonant coupling of the two resonators. For the second sample, we show that the storage resonator can be addressed through a swapping procedure exploiting the resonant coupling of the two resonators.

I. SYSTEM AND MODEL

A. The measured system

The system under study is composed of two microwave resonators (see Fig. 1). The resonators are coupled through a coupling capacitance \( C_c \), permitting the transfer of energy between them. One of the resonators features a tunable resonance frequency. This allows to control the energy exchange between the two resonators, by changing their detuning. The frequency tunability is based on a superconducting quantum interference device (SQUID). It behaves as a tunable, nonlinear, and nondissipative inductance embedded in the resonator.

The tunable resonator has been engineered so that its range of reachable resonance frequency crosses the resonance frequency of the second resonator, which is constant. In addition, the tunable resonator is coupled to a transmission line, which allows us to excite the system and probe it through microwave reflectometry. It will therefore be referred to as the coupling resonator, or resonator B. The other resonator contains no SQUID and thus has a fixed resonance frequency and a long lifetime. It is therefore suitable for microwave storage for instance, and will be referred to as the storage resonator, or resonator A.

B. Theoretical model

The theoretical model of the system is depicted in Fig. 1(a). In the rotating wave approximation, valid because the coupling rate \( g \) between the resonators is much smaller than the resonator resonance frequencies \( \omega_a \) and \( \omega_b \), the Hamiltonian of the coupled resonators is
\[
\hat{H} = \hbar \omega_a \hat{a}^\dagger \hat{a} + \hbar \omega_b \hat{b}^\dagger \hat{b} + \hbar g (\hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b})
+ i \hbar \sqrt{\kappa} \left( V_{in}^\dagger(t) \hat{b} - V_{in}(t) \hat{b}^\dagger \right),
\]
(1)
where \( \hat{a} \) and \( \hat{b} \) are the field ladder operators for resonators A and B, respectively, and \( V_{in}(t) \) the input field driving the system. Note that the Hamiltonian may be time-dependent, as, in addition to the time-dependent drive, the resonance frequency of resonator B \( \omega_b \) can be rapidly tuned in the experiment.

The coupling resonator is capacitively coupled to a transmission line, which makes the system open and dissipative. In addition, both resonators have finite intrinsic lifetimes, \( 1/\kappa_{ia} \) and \( 1/\kappa_{ib} \). To describe the evolution of the quantum state of the system, we use the Lindblad master equation, which gives the time evolution of the density matrix \( \rho = \rho_a \otimes \rho_b \).
\[
\dot{\rho} = -\frac{i}{\hbar} \left[ \hat{H}, \rho \right] + \kappa_{ia} \mathcal{D}[\hat{a}] \rho + \kappa_{ib} \mathcal{D}[\hat{b}] \rho + \kappa g \mathcal{D}[\hat{a}^\dagger \hat{b}] \rho,
\]
(2)
where \( \mathcal{D} \) denotes the Lindblad superoperator, defined as
\[
\mathcal{D}[\hat{x}] \rho = \hat{x} \rho \hat{x}^\dagger - \frac{1}{2} \left\{ \hat{x}^\dagger \hat{x}, \rho \right\}.
\]
Solving this equation gives the time evolution of the average photon number in each resonator, which cannot be measured directly in the experiment. For instance, for the storage resonator,
\[
\langle n_a \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = Tr(\hat{a}^\dagger \hat{a} \rho).
\]
(3)
The classical response of the system to the input field $V_{\text{in}}$ is given by the equations of motion for the expectation values of the reservoir fields $A = \langle \hat{a} \rangle$ and $B = \langle \hat{b} \rangle$. They are derived using $\dot{A} = Tr(\hat{a}\hat{\gamma})$ and $\dot{B} = Tr(\hat{b}\hat{\gamma})$.

\[
\frac{dA}{dt} = -i\omega_A A - igB - \frac{\kappa_{\text{ia}}}{2} A \quad (4)
\]
\[
\frac{dB}{dt} = -i\omega_B B - igA - \frac{\kappa_{\text{ib}}}{2} B - \sqrt{\kappa} V_{\text{in}}. \quad (5)
\]

The output voltage, which can be measured on the transmission line, is computed using the input-output relation

\[V_{\text{out}} = V_{\text{in}} + \sqrt{\kappa} B. \quad (6)\]

In practice, the system is driven at an angular frequency $\omega_d$ close to the resonator resonance frequencies. It is therefore relevant to study its dynamics in a rotating frame. Natural choices for the rotating frame reference frequency are, for instance, the resonance frequency of the storage resonator $\omega_s$, which is constant, or the frequency of the drive field. Redefining the resonator and output fields with respect to a reference frequency are, for instance, the resonance frequency $\omega_{\text{ref}}$, which is constant, or the frequency of the drive field. Redefining the resonator and output fields with respect to a reference frequency $\omega_{\text{ref}}$ by taking $A = ae^{-i\omega_{\text{ref}}t}$, $B = be^{-i\omega_{\text{ref}}t}$ and $V_{\text{out}} = v_{\text{out}}e^{-i\omega_{\text{ref}}t}$, and writing the driving field with respect to the driving frequency $V_{\text{in}} = v_{\text{i}}e^{-i\omega_{\text{d}}t}$, the equations of motion become

\[
\frac{da}{dt} = -i(\omega_a - \omega_{\text{ref}})a - igb - \frac{\kappa_{a}}{2} a \quad (7)
\]
\[
\frac{db}{dt} = -i(\omega_b - \omega_{\text{ref}})b - i ga - \frac{\kappa_{b}}{2} b - \sqrt{\kappa} v_{\text{in}} e^{-i(\omega_d - \omega_{\text{ref}})t}. \quad (8)
\]

We performed simulations of the system by numerically solving either the differential equations of motion, Eq. (7) and (8), or the Lindblad master equation, Eq. (2), using the python package QuTIP\textsuperscript{37,38} dedicated to the study of open quantum systems. In practice, the complex output voltage is measured through heterodyne demodulation of the output signal, which gives its quadrature components $I$ and $Q$. They are defined as $v_{\text{out}} = I + iQ$. In order to directly reproduce the experimental data, the reference frequency of the rotating frame $\omega_{\text{ref}}$ has to be set to the frequency of the local oscillator (LO) used to perform the demodulation (see Fig. 2). The calculated output voltage is subsequently low-pass filtered, to account for the finite bandwidth of the anti-aliasing filter preceding the sampling step in the digitizer. We used a digital Butterworth filter with a cut-off at 90 MHz.

### C. Microwave field oscillation

In order to get an insight to the free evolution of the system, i.e. in absence of a driving field, it is useful to separate the variables $a$ and $b$ in Equations (7) and (8). We choose $\omega_{\text{ref}} = \omega_a$. Neglecting the intrinsic losses ($\kappa_{\text{ia}}, \kappa_{ib} \ll \kappa$, see Table I), this yields, for the storage resonator field,

\[
\frac{d^2a}{dt^2} + \frac{\kappa}{2} i \Delta \frac{da}{dt} + g^2 a = 0. \quad (9)
\]

When the resonators are set on resonance, i.e. $\Delta = \omega_b - \omega_a = 0$, the system behaves as a damped harmonic oscillator characterized by the angular frequency $g$ and the damping ratio $\xi = \kappa/(4g)$. The field in the storage resonator oscillates in time, as the energy is periodically transferred back and forth between the two resonators. It also decays to the transmission line. The decay regime depends on $\xi$. Note that $\xi$ cannot be tuned in situ in the experiment; it is a fixed property of each sample.

For an underdamped system ($\xi < 1$), which corresponds to the experiments shown in this article, the decay is slower than the oscillation, and

\[a(t) = e^{-\xi^2 t} \left( \alpha_1 e^{ig\sqrt{1-\xi^2} t} + \alpha_2 e^{-ig\sqrt{1-\xi^2} t} \right), \quad (10)\]

where $\alpha_{1,2}$ are determined by the initial conditions. The energy, which scales as $|a|^2$, oscillates between the resonators at an angular frequency of $2g\sqrt{1-\xi^2}$. In this strong coupling regime of the two resonators\textsuperscript{39}, the effective coupling rate of the storage resonator to the transmission line is $\kappa_{\text{eff}} = \kappa/2$. It is half of that of the coupling resonator, because, due to the oscillation, the energy is on average only half of the time in the coupling resonator from which it is released to the transmission line. It is only for a critically damped system ($\xi = 1$) that the energy is directly released from the storage resonator to the transmission line at the rate $\kappa/2$ without oscillation. For an overdamped system ($\xi > 1$), corresponding to a weak coupling of the two resonators, the decay is more complex, but it is eventually limited by the coupling rate $\kappa_{\text{eff}} = \frac{1}{2} \left( 1 - \sqrt{1-1/\xi^2} \right)$. For a given $g$, the fastest release of the energy stored in the storage resonator is achieved at the critical damping.

At non-zero detuning, the general solution of Equation (9) is

\[a(t) = \alpha_1 e^{\lambda_+ t} + \alpha_2 e^{\lambda_- t}. \quad (11)\]

Again, $\alpha_{1,2}$ are determined from the initial conditions, and

\[\lambda_{\pm} = -\frac{1}{2} \left( \frac{1}{2} \kappa + i \Delta \right) \pm \frac{1}{2} \sqrt{\left( \frac{1}{2} \kappa + i \Delta \right)^2 - 4g^2}. \quad (12)\]

The two resonance modes of the system are involved in the release process, hence the two terms in the general solution. The real part of $\lambda_{\pm}$ indicates the decay of the field whereas their imaginary part corresponds to the field oscillation.

In underdamped systems, the field oscillation between the two resonators becomes faster when the detuning is increased, but only a decreasing fraction of the energy is
transferred back and forth, so that the storage resonator
is less and less coupled to the transmission line. The first
term of the solution is the most relevant when \( \Delta \gg g \).
The effective coupling rate can be approximated to

\[
\kappa_{\text{eff}} = -2 \Re(\lambda_+ \approx \kappa \frac{g^2}{\Delta^2 + 2g^2}.
\]

(13)

II. DEVICES AND EXPERIMENTAL SETUP

A. Samples

The devices under study are superconducting coplanar
waveguide resonators fabricated on the surface of a silicon wafer. The microwave circuit is primarily made
of niobium. Only the SQUID, which is located in the
middle of the coupling resonator and enables it to be
tunable, is made of aluminum. The fabrication process,
which has already been described elsewhere\(^{33}\), has spe-
cially been designed to obtain as long as possible intrinsic
lifetime for the resonators, with only two electron-beam
lithography steps.

Two samples with two distinct \( \xi \) have been studied.
The parameters of the model described in the previous
section corresponding to these samples are shown in Ta-
ble I. They have been extracted from the different exper-
iments we performed. Note that some results on sample
II have already been presented\(^{33}\).

B. Measurement setup

The measurement setup is displayed in Figure 2. The
sample is kept below 25 mK in a dilution refrigerator,
wired with coaxial lines. The one-port setup at the sam-
ple input is transformed into a two-port measurement
setup using a circulator to route the microwave signals.
This allows to properly attenuate the input signal, which
is necessary for keeping the sample cold and reaching the
few-photon level. The reflected signal is amplified with a
cold 4-8 GHz low-noise amplifier from Low Noise Factory.

The input signal, with an RF frequency up to 6 GHz,
can be arbitrarily modulated both in phase and mag-
nitude with a vector signal generator. The output sig-

FIG. 2. Schematics of the cryogenic microwave measurement
setup. The chip is located at the coldest stage of a dilution
refrigerator equipped with coaxial lines. The input of the de-
vice is probed with a reflectometry setup: a microwave signal
is routed to the device via a circulator and the reflected sig-

nal is amplified with low noise amplifiers, both at 4 K and
room temperature. The output signal is down-converted and
then numerically demodulated and sampled with a vector sig-

nal analyzer. The control port of the chip is driven with
an arbitrary waveform generator (AWG). The input lines are
equipped with attenuators and filters in order to prevent room
temperature thermal noise from heating the device. The out-
put line is equipped with a circulator acting as an isolator.

C. Characterization of the devices with continuous
wave spectroscopy

The resonance modes of the system are probed by ana-
lyzing the reflection coefficient of the devices. Its magni-
tude is shown for sample I in Figure 3(b) as a function of
frequency and the magnetic flux, \( \Phi \), in the SQUID loop.
This experiment is done by measuring the transmission of
a signal between the two ports of the setup with a vector
network analyzer (VNA, not shown in the experimental
setup in Figure 2). The reflection coefficient at the input
of the device is obtained from the raw measurement by
subtracting the part of the signal which comes from the
transmission of the coaxial lines and the microwave com-
ponents. This background is directly measured for half-
integer values of the flux quantum, for which the coupling
resonator is detuned away from the measurement band
and the storage resonator is strongly undercoupled and therefore not seen in the measurement.

1. Resonance mode frequency tuning

Two resonance modes can be seen for each value of the flux. The resonance frequency of the coupling resonator evolves periodically with the flux, because of the periodic modulation of the SQUID inductance and critical current. It follows

\[ \omega_b(\Phi) = \frac{\omega^0_b}{1 + \cos(\pi \frac{\Phi_0}{\Phi})}, \quad (14) \]

where \( \Phi_0 = h/2e \) denotes the flux quantum, \( \omega^0_b \) is its bare resonance frequency (i.e. without the SQUID), and \( \gamma \) is the inductive participation ratio, defined as the ratio of the SQUID inductance at zero flux over the inductance of the coupling resonator. The eigenvalues of the Hamiltonian of the system give the resonance frequencies of the two observed resonance modes.

\[ \omega_{\pm}(\Phi) = \frac{1}{2} \left( \omega_a + \omega_b(\Phi) \right) \pm \sqrt{\frac{g^2}{2} + \left( \frac{\Delta(\Phi)}{2} \right)^2}. \quad (15) \]

They differ from the uncoupled resonance frequencies of the two resonators when they are tuned in resonance, at around \( \pm 0.3 \Phi_0 \) in sample I for instance. A positive detuning is obtained for fluxes around integer number of flux quanta, whereas a negative detuning is obtained around half-integer multiples of flux quanta.

In underdamped samples, the splitting between the two modes (2g) is larger than their widths, which is of the order of \( \kappa/2 \) when the two resonators are on resonance. Therefore, the two modes can be treated separately, as if each mode corresponds to a single resonator mode, with a resonance angular frequency \( \omega_i \), an effective coupling rate \( \kappa_i \) and an effective loss rate \( \kappa_i \). The equations of motion (Eq. (4) and (5)) and the input-output relation (Eq. (6)) can be adapted for such a single resonator, and their resolution in the frequency domain gives the reflection coefficient

\[ \Gamma = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\kappa_{i} - \kappa_{e} - 2i(\omega - \omega_c)}{\kappa_{i} + \kappa_{e} - 2i(\omega - \omega_c)}. \quad (16) \]

Table I contains the parameters for the two samples under study: \( \omega_a \) resonance frequency of the storage resonator, \( \omega^0_b \) bare resonance frequency of the coupling resonator, \( \gamma \) inductive ratio for the coupling resonator, \( g \) coupling between the resonators, \( \kappa \) coupling rate of the coupling resonator to the transmission line, \( \xi \) damping ratio, \( \kappa_{ia} \) dissipation rate of the storage resonator, and \( \kappa_{ib} \) dissipation rate of the coupling resonator.

<table>
<thead>
<tr>
<th>Sample</th>
<th>( \omega_a/2\pi ) GHz</th>
<th>( \omega^0_b/2\pi ) GHz</th>
<th>( \gamma/2\pi ) %</th>
<th>( g/2\pi ) MHz</th>
<th>( \kappa/2\pi ) MHz</th>
<th>( \xi ) MHz</th>
<th>( \kappa_{ia} ) MHz</th>
<th>( \kappa_{ib} ) MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>5.416</td>
<td>5.844</td>
<td>4.8</td>
<td>21.2</td>
<td>5.0</td>
<td>0.0094</td>
<td>0.40</td>
<td>0.125</td>
</tr>
<tr>
<td>II</td>
<td>5.186</td>
<td>5.810</td>
<td>8.4</td>
<td>18.3</td>
<td>280</td>
<td>0.61</td>
<td>0.054</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Fig. 3(a) shows the magnitude and the phase of the reflection coefficient at zero detuning. The two resonance modes are fitted separately with Eq. (16). Note that the phase measured with the VNA follows the electrical engineering rather than the physics convention, thus the substitution \( i \leftrightarrow -j \) had to be done in Eq. (16). Repeating the fitting procedure for every value of the flux allowed us to determine the evolution of the mode reso-
The continuous wave spectroscopy measurements shown in Fig. 4 suggest that the release rate of microwaves stored in the storage resonator can be controlled. To study this, we performed a free decay experiment. In this experiment, the digitizer performing the output signal heterodyne demodulation is set to the frequency of the storage resonator.

Figure 5 shows a comparison of the behavior of samples I (Fig. 5(a)) and II (Fig. 5(b)). Initially, in both cases, the storage resonator is loaded and the coupling resonator is detuned. At \( t = 0 \), the detuning is suddenly reduced. The traces shown corresponds to different final detunings. Overall, both samples show similar behavior: the smaller the detuning the faster the release of the stored energy is.

A noticeable difference is that a much faster release can be achieved with sample II, which simply comes from its larger coupling \( \kappa \) between the coupling resonator and the transmission line. This can also be seen in Fig. 4 through the larger range of the effective coupling \( \kappa_{\text{eff}} \).

The interesting difference lies in the beating that appears for sample I when the detuning is decreased to values of the order of \( g \) or smaller. This occurs because \( \xi < 1 \) for this sample. Many oscillations of the field take place during the release, thus both coupled modes of the system get populated and decay to the transmission line. The beating of the magnitude of the output signal originates from the interference of their two frequencies. The faster beating is observed at zero detuning, where the difference between the frequency of the coupled modes is the smallest. The beating is not observed at too large detunings because one of the two mode is out of the bandwidth of the digitizer. In other words, in the time domain, the beating is faster than the sampling time in this case.

This effect is better seen on the quadratures of the output signal, which clearly show the superposition of two oscillations at two different frequencies. This can be proven by fitting the quadratures with a two-component model

\[
V_{\text{out}}(t) = I(t) + iQ(t) = A_1 e^{-t/\tau_1} e^{-i(\omega_1 t + \phi_1)} + A_2 e^{-t/\tau_2} e^{-i(\omega_2 t + \phi_2)},
\]

as suggested by Equation (11). This has been done for all traces, and the excellent agreement is shown in Fig. 5(c) for \( \Delta = 41 \text{ MHz} \).

III. MICROWAVE SWAPPING

A. Principle

The two-resonator configuration under consideration allows to tune the effective coupling of the storage resonator to the transmission line. This gives the ability to excite the storage resonator from a resonant incoming wave, at a tunable rate. This can be done only at a moderate detuning between the two resonators. The detuning must be larger than \( g \) so that the field builds up only in the storage resonator. It should not be too large.
especially when it is negative, as the resonator should be in the overcoupled regime for the energy transfer to be efficient (see Fig. 4). The range of suitable detuning is nevertheless rather large, in particular for sample II. However, in this mode, the coupling rate can only be smaller than $\kappa/2$, which limits the energy transfer speed.

We now present a different scheme, which utilizes the coupling resonator for transferring microwaves to the storage resonator. Whereas this scheme is only possible in samples with $\xi \ll 1$, typically sample I, it allows both faster energy transfer times and incoming waves with various frequencies (detuned from the storage resonator).

The transfer is realized in two steps (see Fig. 6). Starting with a large detuning, the coupling resonator is first excited by a resonant incoming microwave pulse with a square envelope. Then, when the incoming pulse is turned off, we apply a quick swap operation to transfer the field from the coupling resonator to the storage resonator. This operation is orders of magnitude faster than the decay rate of the coupling resonator, thus a negligible amount of energy is lost instead of being transferred.

The swap operation is realized by bringing the two resonators on resonance. The theory developed in section 1C predicts that a periodic energy transfer between the two resonators should occur. Letting the resonators on resonance for a half integer number of transfer cycles results in a net energy transfer from one resonator to the other.

B. Experimental demonstration

In practice, a 5 $\mu$s-long square microwave pulse at 5.580 GHz is applied at the input of the experimental setup. The power at the input capacitor of sample I is estimated to be -139 dBm, which corresponds to 17 photons in the pulse. This power has been chosen low enough to ensure a linear response of the coupling resonator. At the end of the pulse, the detuning is quickly decreased, kept at zero for 12 ns, and then brought back to its initial value. After a delay of 2 $\mu$s (storage time), the same detuning pulse is repeated. The output signal, measured after heterodyne demodulation and sampling at 200 MS/s, is shown in Fig. 6(c). Its magnitude, its phase, and its quadratures are shown. In this experiment, the digitizer performing the output signal heterodyne demodulation is set to the frequency of the coupling resonator, in contrast with the time domain spectroscopy experiment described in the previous section, where it was set to the frequency of the storage resonator.

The initial rising pattern corresponds to the response of the coupling resonator to the input microwave pulse. Since only a rising exponential pulse can be fully absorbed$^{26}$, a part of the pulse is reflected. Its exact shape depends on the coupling regime ($\kappa/\kappa_a$) of this resonator. At $t = 5 \mu$s, the signal goes quickly towards zero, which proves that most of the energy is removed from the coupling resonator after the detuning pulse. The low-amplitude exponential decay corresponds to the release to the transmission line of energy swapped from the storage resonator, which got excited due to an insufficient detuning during the loading step. What happens when the detuning pulse is applied cannot be probed with the experiment, first because it is too quick to be seen with the detection sampling rate, and more importantly because both resonators are detuned away from the detection bandwidth during the pulse. When the second detuning pulse is applied, the signal shows a fast rise followed by an exponential decay, which proves that this pulse transfers back some energy stored in the storage resonator to the coupling resonator.

The red line in Fig. 6(c) corresponds to a simulation of
FIG. 6. (Color online) Capture and release through field swapping for sample I. (a) Principle of the experiment. A microwave pulse on resonance with the initially detuned (zero flux) coupling resonator is sent to the input port. The detuning is then quickly reduced to zero, leading to a transfer (swapping) of the energy to the storage resonator. After a delay, the swap is repeated, such that the energy is transferred to the coupling resonator from which the microwaves can leak out to the transmission line. (b) Measurement sequence showing the RF input and the detuning. Note that the detuning pulses are 10 times shorter than sketched, for clarity reasons. (c) Magnitude, phase and quadratures of the measured output voltage for an excitation at the input of the resonator of -139 dBm. Red lines: simulation. The phase of the measured trace, which has a random reference, has been compensated to match the phase obtained with the simulation. (d) Same experiment and simulation for an excitation of -119 dBm, driving the coupling resonator to a nonlinear regime. The signal power is a hundred times higher, hence the reduced noise. (e) Color map of the output voltage traces for different duration of the swapping pulses. The swap is effective only for certain pulse widths, where the signal is minimum after the first swap pulse, and maximum after the second pulse. For other values, the energy transfer is only partial.

during the experiment. Equations (7) and (8) are solved using the parameters given in Table I, which were extracted from the continuous wave measurement. The result is superimposed on the measured traces. Only the amplitude of the traces has manually been adjusted, without any physical significance since the system is in the linear regime. The intrinsic loss rate of the coupling resonator, which is difficult to probe in a spectroscopic experiment because it is much lower than its coupling rate, has also been adjusted to obtain the right shape for the output signal in the loading step. The good agreement of the recovered amplitude (after the second detuning pulse) means that the losses, in particular in the storage resonator, are well described by our model. The recovered signal phase is extremely sensitive to the delay between the two pulses, which therefore has been adjusted to ob-
tain the proper distribution of the signal on $I$ and $Q$. The good agreement of the simulation gives us access to new information which cannot be probed in the experiment, for instance the resonator populations.

Fig. 6(d) shows the same experiment performed with an input power of -119 dBm, corresponding to 1700 photons in the pulse. This much higher power drives the coupling resonator to a nonlinear regime, of Duffing type, which arises from the intrinsic nonlinearity of the Josephson junctions of the SQUID. The response to the microwave pulse now shows an oscillating pattern. This behavior can be simulated by adding a nonlinear, cubic term $-ib|b|^2$ to Equation (8). The slight shift of the nonlinear resonator frequency has been accounted for in the simulation, which explains the winding of the phase in the release step.

Figure 6(e) shows that the duration of the detuning pulse must be chosen very precisely. Each horizontal trace corresponds to the output trace of an experiment similar to the one shown in Fig. 6(d), but performed with a variable detuning pulse width, ranging from 0 ns (no detuning pulse) to 150 ns. Note that the delay between the two pulses is only 1 μs. We observe that the detuning pulse alternatively succeeds and fails to transfer the energy between the resonators. This proves that the detuning pulse duration is a full period of this oscillation, the energy simply ends up in the resonator where it was located before the swap pulse. On the other hand, a half-integer number of periods results in swapping the field between the resonators. The behavior of the output voltage at the second detuning pulse is easily understood: when the first pulse keeps the energy in the coupling resonator, it is released to the transmission line directly after, and thus no output signal is observed when the second pulse is applied.

This experiment allows us to determine the optimal detuning pulse width, which is $12 \pm 1$ ns. This is in excellent agreement with the predicted value, given by $2\pi/Aq \approx 11.8$ ns.

IV. CONCLUSION

We designed, fabricated and studied superconducting microwave circuits in which we coupled a superconducting resonator to a transmission line through a frequency-tunable resonator. The detuning between these two resonators can be controlled, which enables to tune the effective coupling of the fixed storage resonator to the transmission line. Moreover, the additional resonance mode introduced by the coupling resonator can get occupied when the two resonators are brought close to resonance. Controlling the detuning therefore also enables the coherent manipulation of microwaves between the two coupled resonators.

The behavior of the system results from the interplay between the oscillation of the field between the coupled resonators and its decay to the transmission line. It depends on a single dimensionless parameter $\xi$ which is the damping ratio of the field oscillation. This parameter can be easily engineered when the circuit is designed by adjusting the coupling capacitances.

This microwave oscillation can be used for catching or releasing microwaves to the transmission line, exploiting the resonant coupling of the two resonators. For samples where $\xi \ll 1$, we showed an efficient strategy for storing microwaves in the system, first loading them into the coupling resonator and then swapping them to the storage resonator. The latter is done by accurately controlling the detuning in time. This strategy allows to catch or release microwaves within a large range of frequencies since the coupling resonator is tunable, while they are stored at a fixed frequency.

In contrast, the off-resonant coupling of the two resonators allows to release (or catch) microwaves only at the frequency of the storage resonator, but with an effective coupling rate which can be tuned. The tuning range is especially large when $\xi$ is not too small. For $\xi \ll 1$, we showed that the two modes of the system are involved in the release process. This could be useful for creating states of the field with a quantum superposition of frequencies.

Numerical simulations of the system showed excellent agreement with the experimental data, demonstrating proper modeling of the system. Whereas at low power, close to the single photon level, linear equations could be used, the modelling is also working at higher photon level by introducing a nonlinear term to account for the nonlinear behavior of the coupling resonator originating from the nonlinearity of the SQUID.

Although we performed experiments with classical signals, it is known that superconducting circuits are suitable for the manipulation of non-classical states of the field. It is likely that the coherent microwaves manipulation which we demonstrated can also be performed with such non-classical states, for instance single photons. The devices described in this article could therefore be implemented as a part of a larger quantum circuit.

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