THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Superconducting qubits - measurement, entanglement, and noise

LARS TORNBERG

Department of Microtechnology and Nanoscience
Applied Quantum Physics Laboratory
Chalmers University of Technology
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LARS TORNBERG

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Chalmers University of Technology
Department of Microtechnology and Nanoscience (MC2)
Applied Quantum Physics Laboratory
SE-412 96 Göteborg, Sweden
Telephone +46 (0)31-772 1000

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ABSTRACT

In the early 1980’s, it was suggested that a computer obeying the laws of quantum mechanics would be able to solve problems beyond the capabilities of a classical computer. The novel ways in which such a quantum computer works relies on the quantum properties of the quantum bits (qubits) used to store the information. Any candidate for a quantum computer must thus be able to sustain these properties and offer means to manipulate and read-out the information. One such candidate are quantum mechanical, superconducting circuits, where the logical bit is encoded in the energy eigenstates of the system. In circuit quantum electrodynamics, such a qubit is coupled to a microwave cavity allowing the qubit to be coherently controlled and read-out by probing the cavity. In this thesis, we theoretically investigate the destructive effects of noise which couple to the system as one tries to measure and control the qubits. We study the so called quantum capacitance read-out scheme, where the state of the qubit is mapped onto an equivalent capacitance of the circuit. It is shown that this is quantum limited, in the sense that the state of the qubit can be determined while simultaneously adding a minimum amount of noise to the system. Apart from the added noise, a measurement on a qubit will perturb it, causing the state to collapse to one of the measurement eigenstates. Such a state collapse can be utilized to generate entanglement between qubits by measuring on the cavity. We show that high-fidelity entangled states can be produced in this way and discuss the potential of the measurement to violate a bound given by local hidden variable theories. The possibility to prolong the life-time of the quantum state by active error correction is also investigated and we discuss limits on gate operation times to benefit from such a code, given realistic values for the error probabilities.

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Keywords: Quantum Computing, Superconducting Qubits, Quantum Electric Circuits, Quantum Measurement, Open Quantum Systems, Quantum Optics
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This thesis is based on the work contained in the following papers,

I Readout methods and devices for Josephson-junction-based solid-state qubits
G. Johansson, L. Tornberg, V. S. Shumeiko and G. Wendin

II Fast quantum limited readout of a superconducting qubit using a slow oscillator
G. Johansson, L. Tornberg and C. M. Wilson

III Dispersive Charge and Flux Qubit Readout as a Quantum Measurement Process
L. Tornberg and G. Johansson

IV Implementation of the three-qubit phase-flip error correction code with superconducting qubits
L. Tornberg, M. Wallquist, G. Johansson, V. S. Shumeiko and G. Wendin

V Randomized Benchmarking and Process Tomography for Gate Errors in a Solid-State Qubit
Phys. Rev. Lett. 102, 090502 (2009)

VI Proposal for generating and detecting multi-qubit GHZ states in circuit QED
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Chapter 1

Introduction

1.1 What are quantum computers?

Quantum computers were first mentioned in 1982 by Richard Feynman in a paper titled *Simulating Physics with Computers* [1]. Here, Feynman posed the question whether physics really could be simulated on a computer, and in that case, what the properties of such a computer would be? Up to that point all computations and simulations of reality had been carried out on either classical, deterministic machines or their probabilistic counterparts. Using simple arguments Feynman showed that it would be impossible for a classical computer of any kind to simulate a realistic, quantum mechanical many-particle system. Instead he proposed to

Let the computer itself be built of quantum mechanical elements which obey quantum mechanical laws.

In 1985 came an important breakthrough when David Deutsch proved the existence of a universal quantum computer [2], the quantum mechanical analog of the classical Turing machine. Deutsch showed that a universal quantum computer could perform any computation that a universal Turing machine could, and moreover, do this in new ways, not possible for any classical machine. In a quantum computer, the logical bits 0 and 1 are encoded into two discrete quantum states of a physical system $|0\rangle$ and $|1\rangle$. It is thus crucial to find a system with only two states or one where such a subset exists, and is efficiently isolated from the remaining states. Such a system is referred to as a quantum bit or qubit, and is the basic building block of a quantum computer. The novel way, in which a quantum computer can operate, is to bring the computational basis (bits) into a superposition of the quantum
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states i.e. $|0\rangle \rightarrow \alpha |0\rangle + \beta |1\rangle$ where $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. This is not equivalent to any classical Turing operation and is the core of all quantum computation. Consider, as an example, two qubits whose evolution is given by the unitary operator $U_f$. (All programs and functions in quantum computation are encoded into unitary operators). In this case $U_f$ encodes the function $f$, which takes the state of first qubit as input and write the result on the second qubit

$$|i,0\rangle \xrightarrow{U_f} |i,f(i)\rangle, \quad i = 0, 1.$$ \hfill (1.1)

By the linearity of quantum evolution, a superposition of input states will then evolve according to

$$\frac{1}{\sqrt{2}} (|0,0\rangle + |1,0\rangle) \xrightarrow{U_f} \frac{1}{\sqrt{2}} (|0,f(0)\rangle + |1,f(1)\rangle), \quad (1.2)$$

thus producing all possible outputs of the function $f$ in one computational step. This is referred to as computation by quantum parallelism [3]. In reality one will work, not with two qubits, but with an entire network of them. Since $N$ qubits amounts to $2^N$ different combinations of 0’s and 1’s this gives rise to a massive parallelism, sometimes reducing the number of computational steps exponentially. The first quantum algorithm which was able to solve a problem intractable for any classical counterpart was presented by Deutsch and Jozsa [3]. The Deutsch-Jozsa algorithm is able to determine a global property of a function $f$ in a single computational step. Later examples are the Grover [4] algorithm for searching unsorted databases, and the Shor [5] algorithm for the discrete logarithm and factorization of large integers. The latter is a good example of a problem insoluble for any classical computer, and has attracted much interest since the security of many modern cryptography techniques relies on the difficulty of this problem.

1.2 Qubits in reality

In 2000, David DiVincenzo gave a list of five important criteria to be fulfilled by any physical realization of a quantum computer [6]. They state that (i) the network of qubits must be scalable, (ii) the qubits must have long coherence times. Furthermore, (iii) it must be possible to control the system so that a universal set of gates can be implemented and (iv) the system must have a qubit-specific measurement capability. The last criterion (v) states that one must be able to initialize an input state. Several suggestions have been cast forward to fulfill these criteria and build a functional quantum
computer. In the qubit family, one distinguishes between two major types of qubits, micro- and macroscopic.

In the microscopic qubits, the computational basis is represented by microscopic degrees of freedom. Quantum information have e.g. been encoded into the photon states of the spatial modes of an electromagnetic field. In the field of nuclear magnetic resonance (NMR), the spin states of atomic nuclei in molecules can be manipulated using radio-frequency electromagnetic waves. Experiments in these systems include the implementation of several small algorithms including a successful factorization of the number 15 using Shor’s algorithm [7]. In ion traps, hyperfine spin states in atoms and the quantized collective motion of their nuclei are used to store and manipulate quantum information. Experiments in these systems are challenging, involving the trapping, and coherent control of atoms with nanometer precision. However, once these technical difficulties are overcome, the qubits exhibit long coherence times allowing for some of the most advanced experiments in quantum information to date, including quantum error correction [8] and the realization of a three-qubit (Toffoli) gate [9].

The macroscopic qubits differ from the microscopic in the respect that the computational basis is encoded in macroscopic quantum states, that are the collective excitations of many microscopic degrees of freedom. Such states exist in superconducting devices where a macroscopic number of electrons form Cooper-pairs and condensate into a single-particle quantum state. The phase of the wave-function plays the role of particle coordinate, and is thus a macroscopic degree of freedom in the above sense. It was long debated whether quantum mechanical effects in macroscopic states could be observed at all, see e.g. Ref. [10]. Because the collective motion of electrons is weakly damped in superconducting systems, they where considered to be the most promising for observing such behaviour. The discovery of macroscopic quantum tunnelling (MQT) of the phase in a current biased Josephson junction [11] was therefore considered as a considerable breakthrough. For a system to be a qubit, it must however exhibit a different, but related property, macroscopic quantum coherence (MQC). This allows two distinctly different macroscopic states to be in a quantum mechanical superposition and to evolve according to the laws of quantum mechanics. The phenomenon of MQC in superconducting systems was predicted in 1985 by Legget [12], but experimentally unverified until 1999 when Nakamura et. al. observed coherent oscillations of Cooper-pairs in a Single Cooper-pair box [13].

The issues of scalability and control do not impose a problem for the macro-
scopic qubits. They can be manufactured on chip and individually addressed using conventional electronics. Here, the major problem is the short coherence time of the encoded quantum state. Because of their huge number of degrees of freedom and the strong Coulomb interaction with the environment, macroscopic quantum systems are extremely sensitive to noise, which destroys the MQC. With better circuit design, such as the development of the Quantronium and transmon qubit (see below), coherence times in superconducting qubits have increased by orders of magnitude since the original experiment by Nakamura. Thus, several groups have in recent years reported successful experiments with coupled superconducting qubits using both fixed [14, 15] and tunable interaction [16, 17]. A two-qubit entangling operation, the controlled-NOT gate has been demonstrated with superconducting flux qubits [18]. This is an important proof of principle, since such an operation is necessary for universal quantum computing.

1.3 Optimal working points for superconducting qubits

In our work we consider mainly the Single Cooper-pair box (SCB) charge qubit, which was mentioned above in relation to the discovery of MQC. This system consists of a superconducting island with coulomb charging energy of an electron, $E_C = e^2/2C$, where $C$ is the capacitance of the island and $e$ is the electron charge. The island is coupled to a reservoir via a Josephson junction, allowing Cooper-pairs to tunnel on and off the island. The rate for the tunneling is given by the Josephson energy $E_J/h$. The two lowest energy eigenstates, which are superpositions of charge eigenstates $|n\rangle$, where $n$ is the number of excess Cooper-pairs on the electrode, are used as qubit states. By tuning a gate voltage, one can control these superpositions as well as the energy difference between the two levels. An important property of the SCB is the existence of an optimal point where the qubit eigenstates are symmetric and anti-symmetric superpositions of the charge eigenstates. At the optimal point the energy bands of the system are flat with respect to gate charge (see Fig. 1.1(a)). Because of this, fluctuations in the gate charge does not induce energy fluctuations in the system which is the source of dephasing. At the same time it makes charge read-out impossible since both qubit eigenstates carry the same average charge. To distinguish the states in a charge measurement, one must therefore move away from the optimal point to a region where the eigenstates are approximately charge states. This in turn makes the qubit extremely sensitive to charge fluctuations in the environment.
Gate charge ($n_0$)

Energy

Figure 1.1: The energy bands of the SCB as a function of gate charge for two choices of $E_J/E_C$-ratio. The optimal point (1) in the $E_J/E_C \ll 1$ limit can be seen in Fig. 1.1(a). Here, fluctuations in $n_0$ does not induce fluctuations in energy as opposed to the situation away from this point (2). In Fig. 1.1(b), we show the transmon limit $E_J/E_C \gg 1$. Here, the lowest lying energy levels are virtually independent of $n_0$ which makes the qubit insensitive to charge noise.

As mentioned above, a major increase in coherence time in superconducting qubits was achieved when Vion et. al. developed a novel measurement scheme for the “quantronium” qubit [19]. In this experiment, the qubit was read out by measuring persistent currents corresponding to the eigenstates, with the qubit parked at an optimal point all the time. The question arose if there was some similar property of the SCB charge qubit which could be exploited to do read-out at the optimal point.

In 2004, the Schoelkopf group [20] managed to couple a charge qubit to a stripline cavity made from a coplanar wave guide, thereby opening up the field of circuit quantum electrodynamics (QED) in solid state electronics. This is in close analogy to cavity QED, where real atoms interact with light confined in a cavity made from high quality mirrors. Such a setup can be used to study fundamental interactions between matter and light. In this field one defines the strong coupling limit as the limit where the interaction strength between atom and cavity is much larger than any coupling of the two to external degrees of freedom. This limit is easily obtained in circuit QED due to the huge dipole moment of the qubit and the strong single-photon field achievable in the wave-guide. With circuit QED, physicists have a new tool for studying non-linear effects in quantum optics.
From the perspective of quantum computing, the interaction between qubit and cavity can be used to perform read-out and control of the qubit state. In the dispersive regime, where the resonance frequency of the cavity $\omega_c$ and the qubit level splitting $\omega_{qb}$ are far de-tuned such that $g/|\omega_c - \omega_{qb}| \ll 1$, where $g$ is the coupling energy between qubit and cavity, the effect of the coupling is to shift the resonance frequency of the cavity, with the sign of the shift depending on the state of the qubit. The state of the qubit can thus be indirectly read out by probing its influence on the cavity. One important point is that this can be done with the qubit at the optimal point. Inspired by this, Duty et. al. embedded an SCB in a resonant LC-circuit and measured its state by probing its effective capacitance [21]. This quantum capacitance is related to the curvature of the qubit energy bands with respect to gate voltage, and was first discussed in the context of small Josephson junctions [22, 23, 24, 25]. At the optimal point the quantum capacitance differs in sign for the qubit eigenstates, which makes it possible to discern them using this property. As in the Schoelkopf setup, this is done by probing the oscillator, whose resonance frequency, and consequently phase-response, depends on the effective capacitance of the circuit.

A natural extension to the idea of optimal working points was made in 2007 with the introduction of the transmon qubit in circuit QED [26]. Here, the Josephson junction is shunted with a relatively large capacitor which makes the energy bands flat such that the optimal “point” is achieved for all values of gate charge as can be seen in Fig. 1.1(b). This makes the transmon relatively insensitive to charge fluctuations and coherence times up to 2 $\mu$s has been measured in these systems. Meanwhile, strong coupling to high-frequency excitations in the resonator is maintained, making it possible to control and read out the qubit through the resonator. Circuit QED with transmon qubits has been very successful. Experimental achievements include coupling qubits using the cavity as a quantum bus [27, 28], generating single photons [29], probing the energy levels of the quantized field in a highly excited cavity, resonantly coupled to a qubit [30], and realization of high fidelity single and two qubit gates [31, 32]. Recently, DiCarlo and co-workers managed to couple two transmon qubits and successfully implement both Grover’s search algorithm and the Deutsch-Josza algorithm in circuit QED [32].
1.4 Entanglement

In this section we introduce the concept of entanglement and discuss its relation to quantum information processing and communication. To start off, we consider the two-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle |12\rangle - |11\rangle |02\rangle),$$

(1.3)

where the subscripts refer to qubit 1 and 2. This state has the property that it cannot be written on the form $|\psi\rangle = |a_1\rangle |b_2\rangle$ where $|a_1\rangle$ and $|b_2\rangle$ are arbitrary single qubit states of qubit 1 and 2. If a state has this property, it is called an entangled state. If a measurement in the qubit eigenbasis is performed on this state, the results are perfectly correlated in the sense that if qubit 1 is found to be in the state $|0\rangle$, the state of qubit 2 can, with certainty, be inferred to be $|1\rangle$ and vice versa. In a famous paper from 1935 [33], Einstein, Podolsky and Rosen (EPR) stated that “If, without in anyway disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.” With this definition, the spin component of qubit 2 clearly is an element of physical reality. However, the state in Eq. (1.3) has the property that a measurement along any quantization axis will give rise to the same correlation as above. ¹ From this, we must infer that all spin components of qubit 2 are elements of physical reality. There is, however, no quantum state of a spin 1/2 particle with definite values for all spin-components. EPR thus conclude that quantum theory must be incomplete if it is assumed that the measurement on qubit 1 does not affect the state of qubit 2, that is if the theory is local. In 1964, Bell published his famous inequality [34], which provides an experimental test to check whether the statistical predictions of a local, complete theory (in the sense of EPR) disagrees with the predictions of quantum mechanics. So far, experimental results are in favor of quantum mechanics, see e.g. Ref. [35]. In 1989, Greenberger, Horne and Zeilinger (GHZ) [36, 37] discovered the entangled state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|01\rangle |02\rangle |03\rangle + |11\rangle |12\rangle |13\rangle),$$

(1.4)

of three particles with the remarkable property that some correlation-experiments on this state must, according to the reasoning of EPR, yield a certain outcome. To see this, consider the measurement of the operator $\sigma_1^y \sigma_2^y \sigma_3^z$, where

¹This can be seen from the fact that $|\psi\rangle$ can be written equivalently as $|\psi\rangle = 1/\sqrt{2}(|c_1\rangle |d_2\rangle - |d_1\rangle |c_2\rangle)$ where $|c\rangle$ and $|d\rangle$ are eigenstates of the operator $\hat{n} \cdot \sigma$.  


Chap. 1: Introduction

the subscripts label the qubits and $\sigma_x$ and $\sigma_y$ are Pauli spin matrices. The state in Eq. (1.4) is an eigenstate of this operator with eigenvalue $-1$. By symmetry, measurement of $\sigma_1^x \sigma_2^x \sigma_3^x$ and $\sigma_1^y \sigma_2^y \sigma_3^y$ will yield the same outcome. Thus, if the spin components of the first two qubits are known, we can infer the state of the third which, according to EPR, makes it an element of physical reality. By symmetry, the same argument holds for the state of the first and second qubit. With this, we can assign definite values to the spin-components of all three qubits $x_i, y_i, i = 1, 2, 3$ prior to the measurement. Since these values do not depend on the choice of measurement local realism gives

$$x_1 x_2 x_3 = (y_1 y_2 x_3)(y_1 x_2 y_3)(x_1 y_2 y_3) = (-1)(-1)(-1) = -1,$$  \hspace{1cm} (1.5)

where we have used that $x_i^2 = y_i^2 = 1$. Quantum mechanics however, gives the result $\sigma_1^x \sigma_2^x \sigma_3^x = 1$ contradicting the result in Eq. (1.5). The refutation of EPR can thus be done in one single experimental run. Using similar reasoning as the above, Mermin showed [38] that the operator

$$M = \sigma_1^x \sigma_2^x \sigma_3^x - \sigma_1^y \sigma_2^y \sigma_3^y - \sigma_1^y \sigma_2^x \sigma_3^y - \sigma_1^x \sigma_2^y \sigma_3^y,$$  \hspace{1cm} (1.6)

satisfies the inequality $M \leq 2$ for any local theory whereas quantum mechanics predicts the value $M = 4$. Moreover, this operator is interesting in the context of circuit QED, since the measurement of $M$ can be conveniently tailored to the dispersive read-out of the qubits through the cavity which was discussed in Sec. 1.3.

Entangled states has been realized in several qubit systems, with the current state of the art being the entanglement of eight qubits in ion traps [39]. Six-qubit GHZ-states have been produced, also in ion traps [40]. In superconducting systems, entangled two-qubit states have been observed and the entanglement verified via state tomography [17]. It is important to show that the system can produce and sustain entanglement since entangled states play a crucial role in quantum teleportation [41], fault-tolerant quantum computing [42] and super-dense coding [43]. Since entanglement is a purely quantum phenomenon it is believed to be one of the novel resources behind the power of quantum computing [42].

1.5 Quantum error-correction

Qubits are very fragile objects, whose quantum properties are easily destroyed because of the interaction with their noisy environment. To realize
a working quantum computer it is therefore imperative to minimize this unwanted effect and a lot of work has gone into various schemes to enhance the qubit life time. One such route has already been presented in Sec. 1.3 where clever circuit design was used to park qubits at optimal points in order to minimize the effect of charge noise. Such a route may be regarded as a sort of passive error correction, where the effect of noise is lowered by reducing the coupling to the noise source. Another strategy is to reduce the strength of the noise itself, which also can be seen as passive error correction. This can be achieved through e.g. efficient filtering of the signals used to read out and control the systems.

A totally different strategy to prolong the qubit life-time is to use active error correction. Here the main idea is to encode the state of the qubit into a larger Hilbert space (code space), $\mathcal{H}_C$, of several physical qubits. The code is constructed such that any error, for which the code can correct, takes the uncorrupted state into a unique subspace of $\mathcal{H}_C$ which is orthogonal to the subspace in which the uncorrupted state resides. The error can be inferred by a measurement, such that appropriate action can be taken to correct for it.

At first glance, quantum error correction might seem impossible since there is a continuum of errors that can occur. One error might flip the state of the qubit, while another adds the relative phase, say, $\pi/34$ to the state. It seems that the bookkeeping of all these errors would require an infinite amount of qubits to encode the state. Remarkably it is possible to show that an arbitrary error can be corrected if the code can correct for the following three errors [42]

\[
\begin{align*}
\alpha |0\rangle + \beta |1\rangle & \rightarrow \alpha |1\rangle + \beta |0\rangle, \quad \text{Bit-flip} \\
\alpha |0\rangle + \beta |1\rangle & \rightarrow \alpha |0\rangle - \beta |1\rangle, \quad \text{Phase-flip} \\
\alpha |0\rangle + \beta |1\rangle & \rightarrow \alpha |1\rangle - \beta |0\rangle, \quad \text{Bitphase-flip},
\end{align*}
\]

which can be seen as the operators $\sigma_x$, $\sigma_z$ and $\sigma_y$ respectively acting on the state. From this, it is easy to see how many qubits are necessary to construct a code which can correct for all single-qubit errors. Let $n$ denote the number of qubits used to encode the state. Given that each qubit can be corrupted in three different ways, there are $1 + 3n$ different states ($3n$ corrupted states plus the uncorrupted), each residing in different orthogonal subspaces in order to be distinguishable. Thus, we require the dimension of $\mathcal{H}_C$ to be at least $2(1 + 3n)$. For $n$ qubits, $\dim \mathcal{H}_C = 2^n$, which gives the inequality $2^n \geq 2(1 + 3n)$. This is violated for all $n \leq 4$ from which we conclude that the minimum number of qubits needed to correct for an arbitrary single qubit
errors is $n = 5$. Several codes such as the nine-qubit Shor code [44], seven qubit Steane code [45] and the five qubit code [46, 47] have been developed that are able to correct for arbitrary errors. We emphasize that all these codes rely on the assumption that errors which simultaneously corrupts two or more qubits can be neglected. With current error-rates in super-conducting systems, this assumption can not be justified. Also from the perspective of these systems, where current state of the art is the coupling of two qubits, the number of qubits required for the above mentioned codes to work make them inaccessible. With this in mind, it is interesting to see what can be achieved with simpler codes that are not able to correct for arbitrary errors. Such codes require less qubits to work and can thus be considered to be within reach of super-conducting implementations. An example of such a code is the three qubit phase flip code which can protect the encoded state from dephasing. Another assumption, which is implicitly made in the discussion of error correction is the ability to do perfect gates. If the gates used to encode the state are faulty, we are not guaranteed that error correction improves the life of the qubit at all. One way to deal with erroneous gates is to do all the initialization, encoding, computation and measurement in a fault-tolerant way [42]. Here, the strategy is to work directly with encoded states comprised of several physical qubits in the way described above. Fault tolerant quantum computing however requires huge overhead in the number of qubits to realize even the simplest computation.

1.6 Quantum efficiency

The theory of quantum measurement states that it is impossible to distinguish the two qubit states faster than the dephasing time [48], after which the qubit has lost its phase coherence. In the weak coupling limit, both phase coherence and measurement uncertainty exhibit an exponential decay in time with rates $\Gamma_\varphi$ and $\Gamma_{ms}$ respectively, satisfying $\Gamma_\varphi \geq \Gamma_{ms}$ [49]. If the only source of coherence loss is the acquisition of information, we say that the measurement is quantum limited with $\Gamma_\varphi = \Gamma_{ms}$. Calculating the ratio between these two rates in terms of circuit parameters therefore gives a good starting point for optimizing the efficiency of the read-out.
1.7 Structure of this thesis

This thesis serves as a background and introduction to the work contained in papers I to VI. In chapter 2 we begin by presenting the treatment of electrical circuits in a quantum mechanical framework. Using this, we derive the Hamiltonian for the single Cooper-pair box and discuss the quantum capacitance. In chapter 3, we use the concepts from chapter 2 to derive the Hamiltonian for a co-planar waveguide resonator. From this we describe photons, thermal- and coherent states which are used to describe the states of the electromagnetic field that are used to read-out and control the qubits in circuit QED. Chapter 4 deals with open quantum systems and we present a derivation of the master equation which describes the evolution of the density matrix when noise is present. Having the master equation, we discuss various error-processes, common in the context of superconducting qubits. Chapter 5 concerns quantum measurements, with emphasis on stochastic master equations which are used to describe the density matrix conditioned on the outcome of a continuous measurement. In this context we also discuss the definition of measurement time and its relation to the quantum efficiency. With this background, we present in Chapter 6 the main results of the work contained in papers I to VI.
Chapter 2

The single Cooper-pair box

In this thesis we consider control and read-out of two types of super-conducting qubits: The single Cooper-pair box (SCB) charge qubit [50] and the transmon qubit [26]. Both the SCB and the transmon consist of a super-conducting island connected to a super-conducting reservoir via a Josephson-junction (JJ), which allows for coherent tunneling of Cooper-pairs on and of the island. In this chapter we describe how such a system can be employed as a qubit.

To only allow transfer of single Cooper-pairs across the JJ, the island must be in the Coulomb blockade regime, and for that, two conditions must be satisfied [51]. Firstly, the tunnel resistance of the junction must be greater than the resistance quantum, \( R_Q = 25.8 \text{ k}\Omega \), which ensures that the Cooper-pairs on the island are localized. Secondly, the charging energy of the island must exceed the temperature \( E_C \gg k_B T \), such that no spontaneous charging of the island can occur. Also, to suppress the tunnelling of quasi-particles across the junction, the energy difference between the even and odd charge states in the superconductor must be larger than the charging energy of the island [52]. This can be achieved by making sure that the super-conducting order parameter, \( \Delta \), is larger than the charging energy \( \Delta \gg E_C \).

The dynamics of the system is governed by two energy scales: the charging energy of the island \( E_C \) and the Josephson tunnelling energy \( E_J \). The difference between the SCB and the transmon is the different values of the ratio \( E_J/E_C \), at which the two systems are operated. For the charge qubit \( E_J/E_C \ll 1 \) and states that correspond to definite charge can be used as a basis for the derivation of the system Hamiltonian. A similar derivation can be done for the transmon, but since this is operated in the \( E_J \gtrsim E_C \).
regime, charge is not a good quantum number which makes the derivation of the Hamiltonian technically more complex. For sake of clarity, we therefore focus on the the SCB operated as a charge qubit. In this limit we derive the two-level Hamiltonian for this system and discuss its operation as a qubit. For a detailed treatment of the transmon, we refer to Ref. [26].

2.1 The Josephson junction

The presence of a zero voltage dc-current (supercurrent) across a barrier between two superconducting leads was predicted in 1962 by Josephson [53] and is known as the Josephson effect. In the leads to the left and right of the junction, the super-conducting condensate is described by a single macroscopic wave-function which we denote $\psi_L = |\psi_L| e^{i\gamma_L}$ and $\psi_R = |\psi_R| e^{i\gamma_R}$ for the left and right lead respectively. The supercurrent is related to the phase difference $\phi = \gamma_R - \gamma_L$ according to

$$I_J = I_C \sin \phi,$$

where $I_C$ is the critical current, which is the maximum supercurrent that the junction can support. If a voltage is applied to the junction, the phase difference becomes time dependent according to

$$\frac{\partial \phi}{\partial t} = \frac{2e}{\hbar} V(t).$$

Eqs. (2.1) and (2.2) are often referred to as the Josephson relations [53]. Apart from the tunneling barrier, the JJ is characterized by a geometric capacitance $C_J$ in parallel with the barrier with the equivalent circuit element for the JJ given in Fig. 2.1. This circuit element will be central to the derivation of the Hamiltonian for the SCB. Before going through this, it

![Figure 2.1:](image)
is therefore instructive to derive the Lagrangian for the circuit in Fig. 2.1. To this end, we consider, as our dynamical variable, the phase-difference \( \phi \) across the JJ as the generalized coordinate of the system. In terms of this, the energy of the Josephson element is given by the integrated electrical work

\[
U_J = \int_{-\infty}^{t} V(t') I(t') dt' = E_J (1 - \cos \phi),
\]

where \( E_J = (\hbar/2e) I_C \) is the Josephson energy of the junction and we have used Eqs. (2.1) and (2.2). Analogous to Eq. (2.2), the phase across the capacitor is defined as \( \phi_C = (2e/\hbar) \int V_C(t) dt \) [54] where \( V_C = -V \) is the voltage across the capacitor. The voltage and current across a capacitor are related to each other by \( I = CV \) where the dot denotes derivative with respect to time. This gives the energy of the junction capacitor

\[
U_C = \frac{C_J V^2}{2} = \left( \frac{\hbar}{2e} \right)^2 \frac{C_J \dot{\phi}^2}{2}.
\]

It is interesting to see that by choosing the superconducting phase as the coordinate we can associate the charging energy of the capacitor with the kinetic energy of the system and, similarly, the Josephson term with the potential energy. With this identification, it is easy to write down the Lagrangian for the system [55] given by the difference between kinetic and potential energies

\[
L_{JJ} = \left( \frac{\hbar}{2e} \right)^2 \frac{C_J \dot{\phi}^2}{2} + E_J \cos \phi,
\]

where the constant term in the potential energy has been dropped since it doesn’t contribute to the dynamics of the system.

### 2.2 The Cooper-pair box Hamiltonian

The the SCB circuit, is shown in figure 2.2(a). A superconducting island is coupled to a lead via a JJ characterized by a junction capacitance \( C_J \) and Josephson energy \( E_J \). An external voltage source \( V_g \) is coupled to the island via a gate capacitance \( C_g \), which is used to externally control the dynamics of the circuit. Following the previous section, we choose the phases across the circuit elements as system coordinates. Using Kirchoff’s voltage law as a constraint, we can write down the Lagrangian solely in terms of the phase across the Josephson junction

\[
L = \frac{C_g(\dot{\phi}_J + V_g)^2}{2} + \frac{C_J \dot{\phi}_J^2}{2} + E_J \cos \phi_J,
\]
Chap. 2: The single Cooper-pair box

(a) The Cooper-Pair box operated as a charge qubit. The charge states of the island act as the computational basis of the qubit. The bits 0 and 1 correspond to the island being occupied by zero or one additional Cooper-pair.

(b) The spectrum of the SCB as a function of the dimensionless gate charge $n_0$. The energy levels for the charge states $n = \{-2, 0, 2\}$ are shown without the Josephson coupling (dashed lines). At the charge degeneracy points $n_0 = -1, 1$, coupling creates a level anti-crossing (solid lines) realizing an isolated two level system.

Figure 2.2: A schematic picture of the SCB together with the circuit energy diagram.

where $\Phi_J = (h/2e)\phi_J$ has the dimension of magnetic flux. To make the transition to a quantum mechanical description of the circuit, we need the Hamiltonian corresponding to the Lagrangian in Eq. (2.6). The Hamiltonian can be obtained from the Lagrangian through a Legendre transformation [55] given by

$$H = \sum_i p_i\dot{x}_i - L,$$

(2.7)

where $x_i$ are the generalized coordinates and the conjugate momenta $p_i$ are defined through

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

(2.8)

Given this, the conjugate momentum to the phase $\Phi_J$ is the charge on the island $p_J = C_\Sigma \Phi_J + C_g V_g$, where $C_\Sigma = C_g + C_J$ is the total capacitance of the island. The Legendre transformation, Eq. (2.7), gives the (classical) circuit Hamiltonian

$$H = E_C(n_J - n_0)^2 - E_J \cos \phi_J,$$

(2.9)

where $E_C = e^2/2C_\Sigma$ is the single electron charging energy of the island. We have defined the dimensionless charge $n_J = p_J/e$ on the island and the di-
mensionless background bias charge \( n_0 = C_g V_g / e \).\footnote{The constant term \(- \frac{C_g V^2}{2}\) has been omitted since it does not affect circuit dynamics.}

Note that so far the Hamiltonian in Eq. (2.9) is given in terms of the classical variables \( n_J \) and \( \phi_J \) or correspondingly \( \Phi_J \) and \( p_J \), satisfying the Poisson bracket relation [55]

\[
\{\Phi_J, p_J\} = 1.
\]

We now quantize the problem through the canonical quantization procedure, which is done by replacing the observables \( \Phi_J \) and \( p_J \), with their corresponding operators \( \hat{\Phi}_J \) and \( \hat{p}_J \) satisfying the commutation relation [56]

\[
[\Phi_J, p_J] = i\hbar,
\]

with the quantum Hamiltonian correspondingly given by

\[
H = E_C (\hat{n}_J - n_0)^2 - E_J \cos \hat{\phi}_J,
\]

where \( \hat{n}_J = \hat{p}_J / e \) and \( \hat{\phi}_J = (2e / \hbar) \hat{\Phi} \). The spectrum of the SCB Hamiltonian as function of gate charge is plotted in Fig. 2.2(b) for the charge states \( |n = -2\rangle \), \( |n = 0\rangle \) and \( |n = 2\rangle \) where \( |n\rangle \) is an eigenstate of the dimensionless charge operator \( \hat{n} |n\rangle = n |n\rangle \). For \( E_J = 0 \) the energy is a quadratic function of \( n_0 \) (dashed lines). At the points \( n_0 = \{-1, 1\} \), the energy of neighboring charge states is degenerate. For \( E_J \neq 0 \), the Josephson term lifts this degeneracy and creates a level anti-crossing, which can be seen in Fig. 2.2(b). These are the optimal points where the system is less sensitive to charge noise as was discussed in Sec. 1.3.

### 2.2.1 Truncating the Hamiltonian

Limiting the gate charge to the neighborhood of \( n_0 = 1 \), and assuming \( E_C \gg E_J \), the two lowest energy levels of the SCB are well-separated from the remaining band structure, see Fig. 2.2(b). In this case, the full Hamiltonian in Eq. (2.9) can be truncated to contain the charge states \( \{0\}, \{2\} \) only. Inserting a complete set of charge states \( |n\rangle \) in Eq. (2.9), together with the observation that the Josephson term can be split into the charge displacement operators

\[
\exp(\pm i\phi_J) |n\rangle = |n \pm 2\rangle
\]

yields the Hamiltonian

\[
H = \sum_n E_C (n - n_0)^2 |n\rangle \langle n| - \frac{E_J}{2} \left( |n + 2\rangle \langle n| + |n - 2\rangle \langle n| \right).
\]
Truncating the Hilbert space as described above gives the reduced Hamiltonian

\[ H = -\frac{E_{el}}{2} \sigma_z - \frac{E_J}{2} \sigma_x, \]  

(2.15)

where the Pauli spin matrices are given by \( \sigma_z = |0\rangle \langle 0| - |2\rangle \langle 2|, \sigma_x = |0\rangle \langle 2| + |2\rangle \langle 0| \). In this way, the qubit can be mapped on a spin-\( \frac{1}{2} \) particle in a magnetic field \( \vec{B} \), with the charging \( E_{el} = \frac{(2e)^2}{2\varepsilon_0 C} (1 - n_0) \) and Josephson energy \( E_J \) acting as the \( z \)- and \( x \)-component of the field respectively \(^2\). The energy eigenstates of the Hamiltonian in Eq. (2.15) are given by

\[ |g\rangle = \cos \frac{\eta}{2} |0\rangle + \sin \frac{\eta}{2} |2\rangle, \]

\[ |e\rangle = \cos \frac{\eta}{2} |2\rangle - \sin \frac{\eta}{2} |0\rangle, \]  

(2.16)

with the angle \( \eta \) given by \( \eta = \arctan \left( \frac{E_J}{E_{el}} \right) \). The states in Eq. (2.16) are used as the computational basis for the charge qubit [57]. In this basis the qubit Hamiltonian is diagonal

\[ H = -\frac{E}{2} \sigma_z, \]  

(2.17)

with the energy splitting \( E \) given by \( E = \sqrt{E_{el}^2 + E_J^2} \).

### 2.3 The quantum capacitance

In paper I, II and III we present a rigorous derivation of the quantum capacitance in a fully quantum mechanical framework. Following Ref. [21], we here present a semi-classical derivation, based on classical electrostatics and the quantization of charge only. This gives the same result as the quantum mechanical derivation, but is more transparent, enabling us to focus on the essential physics.

#### 2.3.1 Semiclassical derivation of the quantum capacitance

Consider the SCB in figure 2.3. The charge on the island can be manipulated using the bias voltage \( V_g \), as was discussed in the previous section. Comparing

\(^2\)The constant offset \( E_C(2 - 2n_0 + n_0^2) \) which do not affect the circuit dynamics have been omitted.
this situation with that of an ordinary capacitance, where the charge is a linear function of the applied voltage $Q = CV$, we can define a differential capacitance of the SCB circuit

$$C_{\text{eff}} = \frac{\partial \langle Q_g \rangle}{\partial V_g},$$

(2.18)

where $Q_g$ is the charge on the gate capacitance $Q_g = C_g(V_g - V_{\text{island}})$, and the brackets denote a quantum mechanical average. With this, the SCB can be modeled as an equivalent linear capacitance $C_{\text{eff}}$, see Fig 2.3. From electrostatics, we get the voltage on the island

$$V_{\text{island}} = \frac{C_g V_g - 2en}{C_{\Sigma}},$$

(2.19)

where $C_{\Sigma} = C_g + C_J$ is the total capacitance of the island and $n$ is the number operator whose eigenvalues correspond to the number of extra Cooper pairs on the island. This gives the charge on the gate capacitor

$$Q_g = \frac{C_J C_g}{C_J + C_g} V_g + 2en \frac{C_g}{C_{\Sigma}}.$$  

(2.20)

To obtain $C_{\text{eff}}$, we must calculate the expectation value of $n$ as a function of $V_g$. In the two-level approximation the number operator is given by

$$n = \frac{1 - \sigma_z}{2}.$$  

(2.21)

With respect to the qubit eigenstates, the number of Cooper pairs will thus be a function of $V_g$. Using Eq. (2.16) it is straightforward to calculate $n$ for the ground $|g\rangle$ and excited $|e\rangle$ qubit states

$$\langle n \rangle^{e/g} = \frac{1}{2} \pm \frac{\cos \eta}{2},$$  

(2.22)
where the angle $\eta$ is given below Eq. (2.16) and the upper and lower sign corresponds to the excited and ground state respectively. In Fig. 2.4(a), we see the number of Cooper-pairs on the island as a function of normalized gate charge. At $n_0 \approx 0$, the electrostatic energy $E_{el}$ is the dominant energy scale

![Diagram](image)

(a) The number of Cooper pairs on the island as a function of $n_0$. (b) The normalized quantum capacitance as function of $n_0$. The sign of $C_Q$ depends on the qubit state.

**Figure 2.4:** Charge and quantum capacitance as functions of normalized gate charge $n_0$. The solid/dashed lines correspond to the ground/excited state.

In the circuit, and the eigenstates are approximately charge states, see also Eq. (2.16). The charge of the island is thus approximately 0 and 1 Cooper pairs for the ground and excited state respectively. For $n_0 \approx 2$ the states have been interchanged so that the ground and excited states now correspond to 1 and 0 Cooper pairs respectively. At the degeneracy point, $n_0 = 1$, the two states have equal charge, making them impossible to distinguish with a charge measurement. The effective capacitance is obtained through the derivative of Eq. (2.22) with respect to $V_g$

$$\frac{\partial \langle n \rangle^{e/g}}{\partial V_g} = \mp \frac{4E_cE_J^2}{\sqrt{16E_c^2(1-ng)^2+E_J^2}}, \quad (2.23)$$

and can be seen in Fig. 2.4(b). Inserting Eq. (2.23) into Eq. (2.18) gives the expression for the effective capacitance at the degeneracy point $n_0 = 1$

$$C_{e/g}^{eff} = \frac{C_J C_g}{C_J + C_g} \mp \frac{2e^2 C_g^2}{E_J C_J C_g}, \quad (2.24)$$

where the first term can be recognized as the total geometric capacitance of the circuit. The second term is due to the coherent tunnelling of Cooper
pairs on and off the island. Thus we define the quantum capacitance at the degeneracy point

\[ C_{e/g} = \pm \frac{2e^2 C_g^2}{E_J C_S^2}. \]  

(2.25)

As seen in Fig. 2.4(b) the magnitude of \( C_{Q} \) is equal for both qubit states, with the sign depending on the state. In the ground/excited state, the capacitance is positive/negative, which is consistent with Fig. 2.4(a). Increasing the voltage in the ground/excited state increases/decreases the charge on \( C_g \).

The definition in Eqn. (2.25) is equivalent to the one given in Ref. [21], where the quantum capacitance is written as the curvature of the energy band

\[ C_{e/g}^{\text{eff}} = \frac{C_J C_g}{C_J + C_g} - \frac{C_g^2}{e^2} \frac{\partial^2 E_{e/g}}{\partial n_g^2}, \]  

(2.26)

with the SCB energy given below Eq. (2.17).
Chapter 3

Circuit quantum electrodynamics

In the previous chapter, we introduced the Josephson junction, described how to construct the single Cooper-pair box and operate it as a charge qubit. The other central part of the system in circuit QED is the cavity or LC-circuit which the qubit is coupled to. In this chapter we show how to derive a harmonic oscillator Hamiltonian from a circuit model of a cavity. Having this, we discuss two important quantum states of this system, which in paper II and III are used to describe measurement back-action. In Sec. 3.4 we show how single qubit gates are implemented in circuit QED by coherently driving the cavity with an appropriate choice of amplitude and phase of the signal.

3.1 The cavity Hamiltonian

The cavity, with capacitance $C_0$ and inductance $L_0$ per unit length, is modeled as a chain of coupled LC-oscillators (see Fig. 3.1). As in Chap. 2, the dynamical variables are chosen to be the fluxes $\Phi_i = (\hbar/2e)\phi_i$ across each capacitor in the chain where $\phi_i$ is related to the voltage drop across the circuit element according to Eq. (2.2). The Lagrangian for the circuit in Fig. 3.1 is given by

$$L = \sum_{i=1}^{N} \frac{C_0\Delta x}{2} \dot{\Phi}_i^2 - \sum_{i=1}^{N-1} \frac{1}{2L_0\Delta x} (\Phi_{i+1} - \Phi_i)^2, \quad (3.1)$$

which in the limit $\Delta x \to 0$ becomes

$$\lim_{\Delta x \to 0} L = \int_0^d dx \frac{C_0}{2} \dot{\Phi}(x,t)^2 - \frac{1}{2L_0} \left( \frac{\partial \Phi(x,t)}{\partial x} \right)^2, \quad (3.2)$$
Figure 3.1: The cavity is modeled as a chain of LC-oscillators with capacitance $C_0$ and inductance $L_0$ per unit length. The dynamical variable is the phase difference $\Phi_i$ across each capacitor.

where $d$ is the length of the cavity. We now seek to eliminate the $x$-dependence from Eq. (3.2) so that the system can be described as a lumped element circuit. The Euler-Lagrange equation [55] for $\Phi(x,t)$ is the wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} - v^2 \frac{\partial^2 \Phi}{\partial x^2} = 0,$$

(3.3)

where $v = 1/\sqrt{L_0 C_0}$ is the wave velocity in the cavity. This has the formal solution

$$\Phi(x,t) = \Phi_1(x + vt) + \Phi_2(x - vt),$$

(3.4)

describing the left and right traveling waves in the cavity. If we assume a cavity where both ends are connected to ground, the boundary conditions read $\Phi(0,t) = \Phi(d,t) = 0$. The solutions to Eq. (3.3) are then given on the form

$$\Phi(x,t) = \Phi \sin(k_n x) \cos(k_n vt),$$

(3.5)

where $k_n = (2n + 1)\pi/(2L)$ and $n = 0, 1, 2, ...$. Picking one of these solutions and inserting that into Eq. (3.2) we can integrate out the $x$-dependence which yields the Lagrangian

$$L = \frac{C_{cav}}{2} \dot{\Phi}(t)^2 - \frac{1}{2L_{cav}} \Phi(t)^2,$$

(3.6)

where $C_{cav} = C_0 d$ and $L_{cav} = 2L_0/(dk_n^2)$. This can be recognized as the Lagrangian for a lumped LC-circuit with effective capacitance $C_{cav}$ and inductance $L_{cav}$. The corresponding Hamiltonian is given by

$$H = \frac{1}{2C_{cav}} p(t)^2 + \frac{1}{2L_{cav}} \Phi(t)^2,$$

(3.7)

where $p(t) = \partial L/\partial \dot{\Phi}$ is the canonical momentum to $\Phi(t)$. The Hamiltonian in Eq. (3.7) is that of a harmonic oscillator with mass $C_{cav}$ and resonance
frequency $\omega = 1/\sqrt{L_{cav}C_{cav}}$. It is convenient to introduce the annihilation operator, $a$, of the cavity such that

$$a = \sqrt{\frac{1}{2\hbar}} \sqrt{\frac{C_{cav}}{L_{cav}}} \left( \Phi + i \sqrt{\frac{L_{cav}}{C_{cav}}} p \right),$$

with the Hamiltonian correspondingly given by

$$H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right).$$

This is the form of the cavity Hamiltonian as presented in most literature where circuit QED is discussed. We see that for each mode, the extended cavity can be modeled as a lumped element $LC$-oscillator whose frequency is that of the given mode. Each mode of the field inside the cavity can be expressed as a linear combination of the creation $a$ and annihilation $a^\dagger$ operators obeying the commutation relation [56]

$$[a, a^\dagger] = 1.$$ 

The action of these operators is to respectively create and annihilate one energy excitation (photon) in the mode

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle,$$

where $|n\rangle$ is an energy eigenstate of the cavity $H |n\rangle = \hbar \omega n |n\rangle$, $n = 0, 1, 2...$

### 3.2 Multi-mode fields

In the previous section, the boundary conditions of the cavity gave rise to a discrete frequency spectrum $\omega_n = \nu k_n$. Assuming that only one mode of the field was excited, the extended cavity was approximated with a lumped element circuit. In Paper II and III we consider the measurement of a qubit coupled to an $LC$-oscillator whose resonance frequency is probed by measuring the phase of a reflected microwave signal, using homodyne detection. Since we are interested in the quantum properties of the signal, the transmission line used to propagate the signal is treated with a similar analysis to that of the cavity in Sec. 3.1. However, since the transmission line is semi-infinite, a continuous set of frequencies is allowed in the system. As opposed to the cavity case, all modes must therefore be taken into account.
and we define the continuum mode creation and annihilation operators $b(\omega)$ and $b^\dagger(\omega)$, satisfying the commutation relation \[58\]

\[ [b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega'). \tag{3.12} \]

As in the case of the single mode field, where $a$ and $a^\dagger$ satisfy Eq. (3.10) to preserve the commutation relation $[\Phi, p] = i\hbar$, the commutation relation in Eqn. (3.12) is chosen to preserve the proper commutation relation between the conjugate phase and charge density fields on the transmission line $[\Phi(x, t), p(x', t)] = i\hbar\delta(x - x')$ where $p(x, t) = C_0\Phi(x, t)$ \[59\].

In paper II and III, we calculate the measurement-induced dephasing rate. As described in detail in chapter 4, this is done by calculating the two-time correlation function of the operator coupling to the qubit. In our case, that operator is the squared charge $p^2(t)$ on the oscillator that couples to the Josephson junction. Since each $p(t)$ contains a linear combination of $b(\omega)$ and $b^\dagger(\omega)$ (see Eq. (3.8)), this expression will contain averages of four field operators e.g.

\[ b(\omega_1)b^\dagger(\omega_2)b^\dagger(\omega_3)b(\omega_4). \tag{3.13} \]

Since field operators of the same frequency do not commute, the ordering of operators is vital, with two orderings being of special importance. In the normal ordering, $N$, all creation operators are positioned to the left of the annihilation operators i.e.

\[ N(aa^\dagger a^\dagger a) = a^\dagger a^\dagger aa. \tag{3.14} \]

as opposed to the anti-normal ordering, $\tilde{N}$, where the all creation operators appear to the right of the annihilation operators. Calculating the average of an operator combination such as that in Eqn. (3.13) with respect to a coherent or thermal state, requires the operators to be on normal form, as we will see in the next section. This can be achieved by moving all the creation operators to the left, using the commutation relation in Eqn. (3.12), or equivalently, to rewrite Eqn. (3.13) on its normal form plus all possible contractions

\[ b(\omega_1)b^\dagger(\omega_2)b^\dagger(\omega_3)b(\omega_4) = \overline{b(\omega_1)b^\dagger(\omega_2)b^\dagger(\omega_3)b(\omega_4)} + \overline{b(\omega_1)b^\dagger(\omega_2)b(\omega_3)b^\dagger(\omega_4)}, \tag{3.15} \]

where the contraction of two field operators is defined to be $\overline{b(\omega_1)b^\dagger(\omega_2)} = \delta(\omega_1 - \omega_2)$. 

Chap. 3: Circuit QED
3.3 Field states

The charge correlation function, \( \langle p^2(t)p^2(t') \rangle \), discussed above is calculated for two cases: During the measurement and in between measurements, where only thermal photons are present in the transmission line. These two cases are represented by different quantum states. In this section, we introduce the two field states, used to model the read-out microwaves and the photons causing thermal dephasing. Given that the operators are on normal form, we show how to evaluate the above field correlation function with respect to these states.

3.3.1 Thermal states

When the photons in the transmission line are thermally excited, we do not have sufficient information to form a pure quantum state of the system. The system is then in a mixed state described by a density operator. Assuming thermal equilibrium at temperature \( T \), this is given by [58]

\[
\rho = \frac{\exp(-\beta H)}{\text{tr}[\exp(-\beta H)]},
\]

where \( H \) is the system Hamiltonian and \( \beta = (k_B T)^{-1} \), where \( k_B \) is Boltzmann’s constant. For a single field mode with frequency \( \omega \) at temperature \( T \), the density matrix can be represented in the photon basis [58]

\[
\rho = \frac{\exp(-\beta \hbar \omega a^\dagger a)}{\text{tr}[\exp(-\beta \hbar \omega a^\dagger a)]} = (1 - \exp(-\beta \hbar \omega)) \exp(-\beta \hbar \omega a^\dagger a),
\]

where we have used that \( \sum_{n=0}^{\infty} x^{-an} = 1/(1 - e^{-a}) \). From this, all moments of the photon number distribution can be calculated. Particularly, the expectation value is given by

\[
\langle n \rangle = \frac{1}{\exp(\beta \hbar \omega) - 1} \equiv \bar{n},
\]

which is the Bose-Einstein occupation number. Since the density matrix in Eq. (3.17) is diagonal in the number basis, the only non-zero moments with respect to \( \rho \) will be those containing an equal number of creation and annihilation operators. We therefore consider the normal ordered expectation values of the number operator \( n = a^\dagger a \),

\[
\langle N(n') \rangle = \langle a^\dagger a' \rangle.
\]
We note that the expectation value in Eqn. (3.19) can be written
\[
\langle a^\dagger a \rangle = \text{tr}[\rho a^\dagger a] = \text{tr}[a^\dagger \rho a],
\]
(3.20)
where we have used the cyclic property of the trace. Hence, if we can find an anti-normal form of \( \rho \), the expectation value in Eqn. (3.19) can be written as the trace of an anti-normal operator. To obtain this we follow Ref. [58] and rewrite the density matrix using the commutation relation in Eqn. (3.10)
\[
\rho = (1 - \exp(-\beta \hbar \omega)) \exp(-\beta \hbar \omega a^\dagger a)
\]
(3.21)
This, together with the ordering theorem given in Ref. [58]
\[
\exp(\theta a^\dagger a) = \tilde{N} \left( \exp \left( [1 - \exp(-\theta)] a^\dagger a \right) \right),
\]
(3.22)
gives the density matrix on anti-normal form
\[
\rho = (\exp(\beta \hbar \omega) - 1) \tilde{N} \left( \exp(-[\exp(\beta \hbar \omega) - 1] a^\dagger a) \right)
\]
(3.23)
The normal moments of the number operator in Eqn. (3.20) can thus be evaluated by calculating the trace of the anti-normal operator
\[
\langle N(n^l) \rangle = \frac{(-1)^l}{n} \frac{\partial^l}{\partial (1/n)^l} \text{tr} \left[ \tilde{N} \left( \exp \left( -\frac{1}{n} a^\dagger a \right) \right) \right]
\]
(3.24)
where the trace has been evaluated by considering the trivial case \( l = 0 \). Intuitively we can understand Eq. (3.24) as how many different combinations of the operator \( n^l \) that can be made from the \( l \) creation and annihilation operators that appear in the trace. For the continuum operators, introduced in Sec. 3.2, the generalization of Eqn. (3.24) is given by
\[
\left\langle \prod_{j=1}^{l} b^l(\omega_j) \prod_{j=l+1}^{2l} b(\omega_j) \right\rangle = \prod_{j=1}^{l} \tilde{n}(\omega_j) \sum_{j=1}^{2l} \prod_{i=1}^{l} \delta(\omega_i - \omega_{i,j}),
\]
(3.25)
where \( \{i, j\} \) is the \( i \)th element of the \( j \)th permutation of the integers \( \{l + 1, \ldots, 2l\} \) [58]. When all the frequencies are equal, Eqn. (3.25) reduces to the single mode result of Eqn. (3.24). As in the single mode case Eq. (3.25) can be understood in terms of the how many different operators \( n(\omega_i) = b^l(\omega_i)b(\omega_j)\delta(\omega_i - \omega_j) \) that can be constructed from the operators in the trace. The delta-functions in Eq. (3.25) simply assures that \( n(\omega_i) \) only counts the photons that belong to the same frequency mode.
3.3.2 Coherent states

Coherent states are the most classical quantum states of the electromagnetic field and thus an appropriate choice to represent a signal generated by an rf-source. They were introduced by Glauber, in his work related to the coherence of light and coincidence measurements of photons [60, 61, 62]. The single-mode coherent state is defined as the displaced vacuum

\[ |\alpha\rangle = D(\alpha)|0\rangle, \]  

where \(D(\alpha)\) is the Glauber displacement operator

\[ D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \]  

where the constant \(\alpha\) is a complex number, with a definite amplitude and phase. We note that \(D(\alpha)\) is unitary with \(D^\dagger(\alpha) = D(-\alpha)\). With the definition in Eqn. (3.26), \( |\alpha\rangle\) is a right eigenstate of the annihilation operator by construction

\[ a|\alpha\rangle = \alpha|\alpha\rangle. \]  

To show this, we use the Baker-Hausdorff relation [56]

\[ e^S A e^{-S} = A + [S, A] + \frac{1}{2} [S, [S, A]] + \ldots, \]  

to calculate the displaced annihilation operator

\[ D^\dagger(\alpha) a D(\alpha) = a + \alpha. \]  

The action of \(a\) on the coherent state is thus given by

\[ a|\alpha\rangle = aD(\alpha)|0\rangle = D(\alpha)D^\dagger(\alpha)aD(\alpha)|0\rangle = D(\alpha)(a + \alpha)|0\rangle = \alpha|\alpha\rangle, \]  

proving Eqn. (3.28). Similarly \(\langle \alpha |\) is a left eigenstate of \(a^\dagger\) with eigenvalue \(\alpha^*\). The continuum analog of the single mode coherent state is obtained by applying the operator \(D[\alpha(\omega)]\) to the vacuum

\[ |\alpha(\omega)\rangle = D[\alpha(\omega)]|0\rangle = \exp \left( \int \alpha(\omega)a^\dagger(\omega) - \alpha(\omega)^*a(\omega)d\omega \right)|0\rangle. \]  

To show that this is a right eigenstate of \(b(\omega')\) with eigenvalue \(\alpha(\omega')\) it is sufficient to show that

\[ D[-\alpha(\omega)]b(\omega')D[\alpha(\omega)] = b(\omega') - \int \left[ \alpha(\omega)b^\dagger(\omega) - \alpha(\omega)^*b(\omega), b(\omega') \right] d\omega = b(\omega') + \alpha(\omega'), \]  

(3.33)
where we have used the commutation relation in Eqn. (3.12). The rest of the proof is identical to that of the single mode case. If the field is in a coherent state it is particularly simple to calculate all normal ordered expectation values. Using that $|\alpha(\omega)\rangle \langle \alpha(\omega)|$ is a right (left) eigenstate of $b(\omega)$ ($b^\dagger(\omega)$), all operators can be replaced with their corresponding classical amplitudes

$$\langle \alpha(\omega)|b^\dagger(\omega_1)b^\dagger(\omega_2)b(\omega_3)b(\omega_4)|\alpha(\omega)\rangle = \alpha^*(\omega_1)\alpha^*(\omega_2)\alpha(\omega_3)\alpha(\omega_4). \quad (3.34)$$

### 3.4 Control of qubits in circuit QED

So far we have considered the charge qubit, which is a SCB operated in the regime $E_C \gg E_J$. In paper IV we investigate the fidelity of single qubit gates where the experiment was carried out on a transmon qubit. The transmon can be understood as an SCB shunted with a large capacitance, which lowers the charging energy of the super-conducting island. Thus, the transmon is operated in the regime $E_J \gtrsim E_C$ which has the effect to lower the sensitivity to low frequency charge noise. In this section, we describe how the state of a transmon-type qubit can be controlled in circuit QED. The starting point is the Hamiltonian for a coupled cavity-transmon system including a coherent drive on the cavity. By applying several unitary transformations, this Hamiltonian can be transformed into an effective Hamiltonian for the driven transmon. For each transformation, we discuss and justify the approximations made.

#### 3.4.1 The cavity - transmon Hamiltonian

The Hamiltonian for the coupled transmon-cavity system is written as [26]

$$H_0 = \omega_r a^\dagger a + \sum_{j=0}^{\infty} \omega_j |j\rangle \langle j| + \sum_{i<j} g_{ij} |i\rangle \langle j| a^\dagger + \text{h.c.}$$

$$+ a\Omega^*(t)e^{i\omega_d t} + a^\dagger\Omega(t)e^{-i\omega_d t}, \quad (3.35)$$

where $\hbar = 1$, $\omega_r$ is the resonance frequency of the cavity and $\omega_j$ is the energy of the $j$th transmon level. The first term in the Hamiltonian can be recognized from Sec. 3.1 as the Hamiltonian of the cavity. The second term describes the uncoupled qubit where the state $|j\rangle$ is the $j$’th energy eigenstate of the transmon Hamiltonian. The third term describes the coupling between one mode of the cavity and the transmon where $g_{ij}$ gives the transition rate from state $|j\rangle$ to $|i\rangle$ due to the coherent exchange of energy between the two
subsystems. The final terms describes the coherent drive of the cavity where the drive amplitude and frequency are given by \( \Omega(t) \) and \( \omega_d \) respectively. We note that the rotating wave approximation [67] has been performed both on the cavity-transmon coupling term and the term in the Hamiltonian that describes the drive on the cavity. This can be done since the drive frequency is resonant with the first transmon transition, \( \omega_d \sim \omega_1 - \omega_0 \). For a thorough introduction to the transmon and the Hamiltonian in Eq. (3.35), we refer to Ref. [26].

### 3.4.2 Transformation to the rotating frame of the drive

We describe the evolution of the system in a frame of reference where the fast oscillations of the drive are eliminated by the time dependent transformation

\[
U = \exp \left[ i \omega_d t \left( a^\dagger a + \sum_j j |j\rangle \langle j| \right) \right].
\]  

(3.36)

Using Eq. (3.29), the Hamiltonian in the rotating frame is given by

\[
\tilde{H}_1 = U H_0 U^\dagger - i U \dot{U}^\dagger = (\omega_r - \omega_d) a^\dagger a + \sum_j (\omega_j - j \omega_d) |j\rangle \langle j| + \sum_{i<j} (g_{ij} |i\rangle \langle j| a^\dagger e^{i \omega_d (i-j+1)t} + \text{h.c.}) + a \Omega^*(t) + a^\dagger \Omega(t),
\]  

(3.37)

where we have used that \( U a U^\dagger = e^{-i \omega_d t} a \) and \( U |i\rangle \langle j| U^\dagger = |j\rangle \langle i| e^{i \omega_d (i-j)t} \). The time dependence in the Hamiltonian has thus been moved from the coherent drive of the cavity to the cavity-transmon coupling. To simplify the equations we will now assume that only nearest neighbor coupling between the transmon levels is relevant, which is known to be valid in the \( E_J/E_C \gg 1 \) limit [26]. With this, the time dependence in the coupling term vanishes and the Hamiltonian in the rotating frame becomes

\[
H_1 = (\omega_r - \omega_d) a^\dagger a + \sum_j (\omega_j - j \omega_d) |j\rangle \langle j| + \sum_j (g_{j,j+1} |j\rangle \langle j+1| a^\dagger + \text{h.c.}) + a \Omega^*(t) + a^\dagger \Omega(t).
\]  

(3.38)
3.4.3 Displacement transformation

Since the goal is to eliminate the cavity degrees of freedom from the Hamiltonian, we want to transfer the drive from the cavity to the transmon. This is done with the time dependent displacement transformation

\[ D = \exp[\alpha(t)a \dagger - \alpha^*(t)a], \quad (3.39) \]

such that \( D \dagger a D = a + \alpha(t) \) as in Eq. (3.30). This gives the Hamiltonian

\[ \tilde{H}_2 = (\omega_r - \omega_d)a \dagger a + \sum_j (\omega_j - j\omega_d)|j\rangle\langle j| \]

\[ + \sum_j [g_{j,j+1}|j\rangle\langle j+1|\alpha^*(t) + \text{h.c.}] + \sum_j [g_{j,j+1}|j\rangle\langle j+1|a \dagger + \text{h.c.}] \]

\[ + (a\Omega^*(t) + a \dagger \Omega(t)) + (\omega_r - \omega_d)(\alpha(t)a \dagger + \alpha^*(t)a) - i[\dot{\alpha}(t)a \dagger - \text{h.c.}], \]

where we now choose the function \( \alpha(t) \) to satisfy the differential equation

\[-i\dot{\alpha}(t) + (\omega_r - \omega_d)\alpha(t) + \Omega(t) = 0. \quad (3.41)\]

Given that \( \alpha(t) \) satisfies Eq. (3.41), the terms in the last line of Eq. (3.40) cancel and the Hamiltonian in the displaced frame of reference becomes

\[ H_2 = (\omega_r - \omega_d)a \dagger a + \sum_j (\omega_j - j\omega_d)|j\rangle\langle j| \]

\[ + \sum_j [g_{j,j+1}|j\rangle\langle j+1|\alpha^*(t) + \text{h.c.}] + \sum_j [g_{j,j+1}|j\rangle\langle j+1|a \dagger + \text{h.c.}], \]

where the drive now induces transitions between the levels in the transmon.

3.4.4 Dispersive transformation

The Hamiltonian in Eq. (3.42) can in principle be used to study and simulate the driven system. We can however make one final transformation to eliminate the cavity degrees of freedom from the dynamics, which greatly simplifies numerical simulations. If the cavity and transmon are sufficiently detuned such that, \( g_{j,j+1}/\Delta_{j,j+1} \ll 1 \) where \( \Delta_{j,j+1} = \omega_{j+1} - \omega_j - \omega_r \), we make the unitary transformation

\[ H_D = U_D H_2 U_D^\dagger, \quad (3.43) \]

with

\[ U_D = \exp \left( \sum_j \lambda_{j,j+1}|j\rangle\langle j+1| - \text{h.c.} \right), \quad (3.44) \]
and expand to second order in the small parameter $\lambda_j = -g_{j,j+1}/\Delta_{j,j+1}$. This gives the following transformed terms

\[
U_D a U_D^\dagger = a - \sum_j \lambda_j |j\rangle \langle j + 1|,
\]

\[
U_D a^\dagger a U_D^\dagger = a^\dagger a - \sum_j \lambda_j |j\rangle \langle j + 1| a^\dagger - \sum_j \lambda_j |j + 1\rangle \langle j| a^\dagger a + \sum_j \lambda_j^2 |j + 1\rangle \langle j + 1| a^\dagger a,
\]

\[
U_D |j\rangle \langle j| U_D^\dagger = |j\rangle \langle j| + \left[ (\lambda_{j-1} |j - 1\rangle \langle j| - \lambda_j |j\rangle \langle j + 1|) a^\dagger + \text{h.c.} \right] + \left[ \lambda_{j-1}^2 |j\rangle \langle j| - \lambda_j^2 |j + 1\rangle \langle j + 1| a^\dagger \right] + \left[ \lambda_j^2 |j\rangle \langle j - 1| - \lambda_{j+1}^2 |j - 1\rangle \langle j - 1| a^\dagger a, \right]
\]

\[
U_D |j\rangle \langle j + 1| U_D^\dagger = |j\rangle \langle j + 1| + \lambda_{j-1} |j - 1\rangle \langle j + 1| a^\dagger - \lambda_j |j + 1\rangle \langle j + 1| a - \lambda_{j+1} |j\rangle \langle j + 2| a^\dagger + \lambda_j |j\rangle \langle j| a,
\]

where we have thrown away rotating terms that are second order in $\lambda_i$. Inserting Eq. (3.45) into Eq. (3.42) gives the final Hamiltonian in the rotating, displaced and dispersive frame

\[
H_D = (\omega_r - \omega_d) a^\dagger a + \sum_j (\omega_j' - j\omega_d) |j\rangle \langle j| + \sum_j [g_{j,j+1}|j\rangle \langle j + 1| a^\dagger (t) + \text{h.c.}] + \sum_j (\chi_{j-1,j} - \chi_{j,j+1}) a^\dagger a |j\rangle \langle j|
\]

where, once more, rotating terms proportional to $\lambda_i^2$ have been discarded. Here, we have defined $\omega_j' = \omega_j + \chi_{j-1,j}$ and the residual cavity-transmon coupling $\chi_{j,j+1} = g_{j,j+1}^2/\Delta_{j,j+1}$. In the dispersive regime the effect of the transmon-cavity coupling is two-fold, as can be seen in Eq. (3.46). First, the resonance frequencies of the transmon get shifted due to the Lamb shift $\omega_j \to \omega_j'$. Second, the cavity experiences an ac-Stark shift (last line in Eq. (3.46)) which gives a qubit dependent shift of the cavity resonance frequency. The dynamics induced by the Hamiltonian in Eq. (3.46) can by used to realize single qubits gates, as is explained in the remainder of this section.
3.4.5 Single qubit gates

Since the drive is resonant with the first transmon transition frequency and thus very detuned from the cavity, we can safely neglect the cavity occupation number and put $a^\dagger a = 0$. Ideally, the coherent evolution due to the drive should only affect the ground and first excited state of the transmon. Thus we truncate the transmon Hilbert space to the two lowest energy eigenstates, approximating the Hamiltonian in Eq. (3.46) by

$$H_D = g_{0,1} \{ \sigma_- \alpha^*(t) + \sigma_+ \alpha(t) \}, \quad (3.47)$$

where we have defined the lowering operator for the two levels system $\sigma_- = |0\rangle \langle 1|$ and correspondingly $\sigma_+ = (\sigma_-)^\dagger$. Depending on the phase of the drive, $\alpha(t)$ can be made purely real or imaginary corresponding to the two cases

$$H_D = g_{0,1} |\alpha(t)| \sigma_x, \quad \alpha(t) \text{ real},$$
$$H_D = g_{0,1} |\alpha(t)| \sigma_y, \quad \alpha(t) \text{ imaginary}. \quad (3.48)$$

Thus, depending on the phase of the signal the coherent evolution of the qubit state corresponds to rotations $R_x(\theta)$ or $R_y(\theta)$ around the $x$ and $y$-axis on the Bloch-sphere with the rotation angle determined by $\theta/2 = \int dt g_{01} |\alpha(t)|$ [42].
Chapter 4

Errors and noise

If I do it all for love, will I ever give enough?
Cuz you can never be too pure or too connected

Sebadoh: Too Pure

In this chapter we review the treatment of noise in quantum systems, focusing on the derivation of a master equation from underlying Hamiltonian dynamics [63]. We show what approximations are needed in order to get such an equation and discuss on what physical grounds these approximations can be made. In Sec. 4.2 the derived master equation is used to describe the dynamics of a quantum two-level system (qubit) coupled to a bath allowing energy to dissipate from the system. In Sec. 4.5 we emphasize the dependence of energy eigen-basis when deriving the master equation.

4.1 Derivation of the master equation

We consider a system $S$ coupled to a bath $B$ where the Hamiltonian for the total system is given by

$$H = H_S + H_B + V,$$  \hspace{1cm} (4.1)

where the coupling Hamiltonian $V$ is assumed to be small compared to $H_S + H_B$. Let $\sigma(t)$ be the density matrix of the combined system-bath Hilbert space such that the Liouville equation for $\sigma(t)$ reads

$$i\dot{\sigma} = [H_S + H_B + V, \sigma],$$  \hspace{1cm} (4.2)

where we have set $\hbar = 1$. Since we are interested in the slow dynamics given by $V$, it is convenient to describe the evolution in the interaction picture of
with
\[ U_0(t) = e^{-i(H_S + H_B)t}, \] (4.4)
so that the Liouville equation takes the form
\[ i\dot{\sigma}_I = [V_I(t), \sigma_I(t)], \] (4.5)
where \( V_I(t) = U_0^\dagger(t) V U_0(t) \) is the interaction Hamiltonian in the interaction picture. Writing Eq. (4.5) on integral form
\[ \sigma_I(t) = \sigma_I(0) - i \int_0^t dt' [V_I(t'), \sigma_I(t')], \] (4.6)
and substituting back into Eq. (4.5) yields
\[ \dot{\sigma}_I = -i[V_I(t), \sigma_I(0)] - \int_0^t dt' [V_I(t), [V_I(t'), \sigma_I(t')]]. \] (4.7)
Eq. (4.7) still contains the full density matrix \( \sigma_I(t) \) on the right hand side. If the coupling between the system and bath is sufficiently weak we may assume that the interaction between the system and bath has a negligible influence on the latter. This is justified if the size of the bath is much larger than the size of the system in which case we can assume that the fractional effect of the interaction is much larger on the system. In that case, we may write the density matrix as the tensor product
\[ \sigma_I(t) \approx \rho_I(t) \otimes \rho_B \] (4.8)
where \( \rho_I(t) \) and \( \rho_B \) are the density matrices of the system and bath respectively. Since the bath is assumed to be unaffected by the interaction, we may regard \( \rho_B \) to be constant in time. Substituting Eq. (4.8) into Eq. (4.7) and tracing over the bath yields an integro-differential equation for the system density matrix
\[ \dot{\rho}_I = -\text{tr}_B \int_0^t dt' [V_I(t), [V_I(t'), \rho_I(t') \otimes \rho_B]], \] (4.9)
where we have assumed that \( \text{tr}_B([V_I(t), \sigma_I(0)]) = 0 \). Eq. (4.9) is not yet on Markovian form since the time evolution at time \( t \) may depend on the state of the system at all previous times \( t' < t \). To turn Eq. (4.9) into a Markovian master equation, we assume that the interaction Hamiltonian can be written
as a sum of terms like $A_S(t)X_B(t)$ where the operators $A_S(t)$ and $X_B(t)$ act on the system and bath degrees of freedom respectively. The integrand in Eq. (4.9) will then contain correlation functions given by

$$\langle X(t)X(t') \rangle = \text{tr}_B(X(t)X(t')\rho_B),$$

(4.10)

which determine the correlation time $\tau_B$ of the bath, such that $\langle X(t)X(t') \rangle \approx 0$ for $t - t' > \tau_B$. Since the rate of change of the density matrix in the interaction picture is given by the coupling to the bath, while the time evolution of the operators in Eq. (4.10) is set by $H_B$ we may assume the bath correlation time $\tau_B$ to be much shorter than any time for which $\rho_I(t)$ varies considerably. In this case we can substitute $t' = t - \tau$ and let the upper limit of the integral in Eq. (4.9) go to infinity. Moreover we may substitute $\rho_I(t') \rightarrow \rho_I(t)$. This is known as the Markov approximation since the resulting equation is first order differential equation

$$\dot{\rho}_I = -\text{tr}_B \left( \int_0^\infty d\tau \, [V_I(t), [V_I(t - \tau), \rho_I(t) \otimes \rho_B]] \right),$$

(4.11)

such that the knowledge about the state at time $t_0$ gives the state at all times $t > t_0$.

We will now consider a specific type of interaction Hamiltonian where the bath is taken to be a collection of harmonic oscillators. Since the electromagnetic field is known to be equivalent to such a system, this type of environment is highly relevant for qubits in solid state systems. Thus we consider $V(t)$ above to be on the form

$$V_I(t) = A_I(t) \sum_{i=0}^\infty \kappa_i \left( b_i(t) + b_i^\dagger(t) \right) = A_I(t)X_I(t),$$

(4.12)

where $A_I(t)$ is a Hermitian operator acting on the system Hilbert space and $b_i(t)$ is the annihilation operator of mode $i$ in the interaction picture and $\kappa_i$ is the coupling between the system and mode $i$. The bath Hamiltonian is given by $H_B = \sum_i \omega_i b_i^\dagger b_i$. Evaluating the double commutator in the integral we obtain

$$\dot{\rho}_I = \int_0^\infty d\tau \text{tr}_B \left[ V_I(t - \tau)\rho_I(t)\rho_B - V_I(t)\rho_I(t - \tau)\rho_B \right] + \text{h.c.}$$

(4.13)

Performing the partial trace over the bath degrees of freedom now gives the master equation

$$\dot{\rho}_I = \int_0^\infty d\tau \langle X_I(t)X_I(t - \tau) \rangle \times \left( A_I(t - \tau)\rho_I(t)A_I(t) - A_I(t)A_I(t - \tau)\rho_I(t) \right) + \text{h.c.}$$

(4.14)
The time dependence of the operator $A_I(t)$ is in general complicated and we therefore project $A_I(t)$ on to the energy eigenstates $\{ |m\rangle \}$ of the Hamiltonian $H_S$ such that

$$A_I(t) = \sum_{mn} |m\rangle \langle m| A_I |n\rangle \langle n| e^{i\omega_{mn} t} \equiv \sum_{mn} A_{mn} |m\rangle \langle n| e^{i\omega_{mn} t} \quad (4.15)$$

where $\omega_{mn} = \omega_m - \omega_n$, and $\hbar \omega_m$ is the energy of the state $|m\rangle$. Inserting this expression into Eq. (4.14) we obtain

$$\dot{\rho}_I = \sum_{mn} \sum_{m'n'} \Gamma(\omega_{mn}) A_{mn}^* A_{m'n'} e^{i(\omega_{m'n'} - \omega_{mn}) t} \times \left( |n\rangle \langle m| \rho_I(t) |m'\rangle \langle n'| - |m'\rangle \langle n'| |n\rangle \langle m| \rho_I(t) \right) + \text{h.c.} \quad (4.16)$$

where we have defined the one sided Fourier transform of the bath correlator

$$\Gamma(\omega_{mn}) = \int_0^\infty d\tau \langle X_I(t) X_I(t - \tau) \rangle e^{i\omega_{mn} \tau}. \quad (4.17)$$

Assuming that the bath is in thermal equilibrium, implying that $[H_B, \rho_B] = [U_0(t), \rho_B] = 0$, the bath correlator is homogenous in time

$$\langle X_I(t) X_I(t - \tau) \rangle = \text{Tr}_B \left( U(t) X U(t) U(t - \tau) X U(t - \tau) \rho_0 \right) = \text{Tr}_B \left( U(t) X U(t) \rho_0 \right) = \text{Tr}_B \left( X_I(\tau) X_I(0) \rho_0 \right) = \langle X_I(\tau) X_I(0) \rangle \quad (4.18)$$

which makes $\Gamma(\omega_{mn})$ time independent.

4.1.1 The secular approximation

It is possible to simplify Eq. (4.16) even further by separating out, and discarding terms that oscillate fast on the time-scale of the system evolution in the interaction picture. This is the secular approximation which we will now describe. To this end we define two separate time scales of the system evolution, $\tau_S$ and $\tau_I$. The time scale $\tau_S$ is the typical time over which the system evolves due to its own Hamiltonian $H_S$. This is set by the inverse of the typical energy differences in the system $\tau_S \sim |\omega_{mn} - \omega_{m'n'}|^{-1}$. The other time scale $\tau_I$ is similarly given by the inverse coupling strength to the bath $\tau_I \sim \kappa^{-1}$. Since our starting point is the weak coupling limit, it is clear that $\tau_S \ll \tau_I$. Hence, all terms in Eq. (4.16) with $|\omega_{mn} - \omega_{m'n'}| \neq 0$ will oscillate
rapidly on the time scale of the slowly evolving system in the interaction picture and average to zero. Assuming that we have no degeneracies in the system we thus only keep terms for which \( m = m' \) and \( n = n' \) and terms such that \( m = n \) and \( m' = n' \). In the secular approximation, Eq. (4.16) is then written

\[
\dot{\rho}_I = \sum_{mn} \Gamma(\omega_{mn}) A_{mn}^* A_{mn} \left( |n\rangle \langle m| \rho_I(t) |n\rangle \langle m| - |m\rangle \langle n| \langle m| \rho_I(t) \right) + \text{h.c.}
\]  

(4.19)

Inserting this into Eq. (4.19) we finally arrive at the master equation for the reduced density matrix of the system

\[
\dot{\rho}_I = -i[H_{ls}, \rho_I] + D[\rho_I],
\]  

(4.21)

from which we can distinguish two different types of evolution due to the interaction with the bath. The first term in Eq. (4.21) represents a Hamiltonian type of evolution where the Hermitian operator \( H_{ls} = \sum_{mn} \Im[\Gamma(\omega_{mn})] A_{mn}^* A_{mn} |m\rangle \langle m| + \sum_n \Im[\Gamma(\omega_{nn})] A_{nn}^* A_{nn} |n\rangle \langle n| \) can be added to \( H_S \) in the Schrödinger picture. In optical systems, this term is called the Lamb shift and gives rise to re-normalization of the unperturbed energy levels. The second term in Eq. (4.21) is given by the super-operator

\[
D[\rho_I] = \sum_{mn} 2\Re[\Gamma(\omega_{mn})] A_{mn}^* A_{mn} \left[ |n\rangle \langle m| \rho_I(t) |n\rangle \langle m| - \frac{1}{2} \left\{ |m\rangle \langle n| \langle n| \langle m|, \rho_I(t) \right\} \right] + \text{h.c.}
\]  

(4.23)

which represents the part of the master equation that induces irreversible dynamics in the system. Because of this, \( D[\rho] \) is often called the dissipator of the system. The strength of the dissipation is given by the real part of \( \Gamma(\omega) \) at the transition frequencies \( \omega = \omega_{mn} \) corresponding to the transitions between the states \( |m\rangle \) and \( |n\rangle \) induced by the operator \( A \). Given the interaction hamiltonian in Eq. (4.12) the bath correlator is given by (see Eq. (3.25))

\[
\langle X_I(\tau) X_I(0) \rangle = \sum_i \kappa_i^2 \left[ (\bar{n}(\omega_i, T) + 1) e^{-i\omega_i \tau} + \bar{n}(\omega_i, T) e^{i\omega_i \tau} \right],
\]  

(4.24)
where $\bar{n}(\omega, T) = [\exp(\hbar \omega / k_B T) - 1]^{-1}$ is the Bose occupation number reflecting the bosonic nature of the bath. Furthermore, we consider the frequency spectrum of the bath in the continuum limit, allowing us to replace the sum in Eq. (4.24) with an integral

$$\sum_i \rightarrow \int_0^\infty g(\omega),$$

(4.25)

where $g(\omega)$ is the mode density of states. Inserting Eq. (4.24) and Eq. (4.25) into Eq. (4.17) the one-sided Fourier transform of the bath correlation function is given by

$$\Gamma(\omega_{mn}) = \int_0^\infty d\tau \int_0^\infty d\omega g(\omega) \kappa(\omega)^2 \left[ (\bar{n}(\omega, T) + 1) e^{i(\omega_{mn} - \omega)\tau} + \bar{n}(\omega, T) e^{i(\omega_{mn} + \omega)\tau} \right],$$

(4.26)

where the real and imaginary parts are given by

$$\Re[\Gamma(\omega_{mn})] = \pi \int_0^\infty d\omega g(\omega) \kappa(\omega)^2 \times \left[ (\bar{n}(\omega, T) + 1) \delta(\omega - \omega_{mn}) + \bar{n}(\omega, T) \delta(\omega + \omega_{mn}) \right]$$

and

$$\Im[\Gamma(\omega_{mn})] = \mathcal{P} \int_0^\infty d\omega g(\omega) \kappa(\omega)^2 \left[ \frac{\bar{n}(\omega, T) + 1}{\omega_{mn} - \omega} + \frac{\bar{n}(\omega, T)}{\omega + \omega_{mn}} \right],$$

(4.28)

respectively. Here $\mathcal{P}$ denotes the Cauchy principal value which can be reviewed in Appendix B.

### 4.2 An example: relaxation of a two-level system

The master equation in Eq. (4.21) is very versatile in the sense that we haven’t specified the system Hamiltonian $H_S$ nor the system operator $A$ appearing in the interaction part $V$. In this section we consider a qubit whose Hamiltonian is given by

$$H_S = -\frac{\omega_q h}{2} \sigma_z,$$

(4.29)

coupled to a bath with the interaction Hamiltonian

$$V_i = \sigma_x(t) X(t).$$

(4.30)
The coupling in Eq. (4.30) is called transversal since \( \sigma_x \) is perpendicular to the chosen quantization axis (\( \sigma_z \)) for the qubit. In Sec. 4.3.1, we solve the obtained master equation and show that the type of coupling above gives rise to energy dissipation. Choosing the system Hamiltonian as in Eq. (4.29), the operator, as defined in Eq. (4.15) is given by

\[
A(t) = |0\rangle\langle 1|e^{i\omega_{10}t} + |1\rangle\langle 0|e^{i\omega_{10}t},
\]

where \( \omega_{10} = -\omega_{01} = \omega_{qb} \). Inserting this into Eq. (4.23) and (4.22) we see that the all diagonal terms vanish yielding the dissipator

\[
D[\rho_I] = 2\Re[\Gamma(\omega_{01})] \left[ |1\rangle\langle 0| \rho_I(t) |0\rangle\langle 1| - \frac{1}{2} \{ |0\rangle\langle 1|, \rho_I(t) \} \right] + 2\Re[\Gamma(\omega_{10})] \left[ |0\rangle\langle 1| \rho_I(t) |1\rangle\langle 0| - \frac{1}{2} \{ |1\rangle\langle 0|, \rho_I(t) \} \right]
\]

\[
= \gamma\bar{n}(\omega_{qb}, T) \left( \sigma_+ \rho \sigma_+ - \frac{1}{2} \{ \sigma_+, \rho \} \right) + \gamma \left( \bar{n}(\omega_{qb}, T) + 1 \right) \left( \sigma_- \rho \sigma_- - \frac{1}{2} \{ \sigma_-, \rho \} \right)
\]

(4.32)

where \( \gamma = 2\pi g(\omega_{qb})|\kappa(\omega_{qb})|^2 \) and we have defined the raising- \( \sigma_+ = |1\rangle \langle 0| \) and lowering operator \( \sigma_- = |0\rangle \langle 1| \). The hermitian operator adding to the unitary evolution is given by

\[
H_{ls} = \Im[\Gamma(\omega_{01})]|0\rangle\langle 0| + \Im[\Gamma(\omega_{10})]|1\rangle\langle 1|
\]

\[
= \frac{1}{2} (\Im[\Gamma(\omega_{01})] - \Im[\Gamma(\omega_{10})]) \sigma_z
\]

\[
= -\mathcal{P} \int_0^\infty d\omega g(\omega) \kappa(\omega)^2 (2\bar{n}(\omega, T) + 1) \left( \frac{1}{\omega_{qb} - \omega} + \frac{1}{\omega_{qb} + \omega} \right) \sigma_z,
\]

where we have used that \( |0\rangle \langle 0| = (1 + \sigma_z)/2 \) and \( |1\rangle \langle 1| = (1 - \sigma_z)/2 \). We note that the dissipator in Eq. (4.32) is on the form

\[
D[\rho] = \sum_j L_j \rho L_j^\dagger - \frac{1}{2} \{ L_j^\dagger L_j, \rho \},
\]

(4.34)

with \( L_1 = \sqrt{\gamma\bar{n}(\omega_{qb}, T)} \sigma_+ \) and \( L_2 = \sqrt{\gamma \left( \bar{n}(\omega_{qb}, T) + 1 \right)} \sigma_- \). This is known as the Lindblad form [64] where the structure of \( D \) guarantees that the master equation preserves trace and positivity of the density matrix.
4.3 Relaxation and dephasing

In this section we describe two different types of errors, commonly encountered in the context of superconducting qubits; relaxation and dephasing. The master equation for relaxation was derived in Sec. 4.2 and the corresponding equation for dephasing will be given in Sec. 4.3.2. For both errors we solve the master equation to see the effect of the bath on the qubit density matrix.

4.3.1 Relaxation

In Sec. 4.2, we derived the master equation for a qubit coupled transversally to a bath of harmonic oscillators. Ignoring the Lamb shift, we may re-write Eq. (4.32) in the form of a coupled system of equations for the components of the density matrix

\[
\begin{align*}
\dot{\rho}_{00}(t) &= -\Gamma_1 \rho_{00}(t) + \Gamma_1 \rho_{11}(t) \\
\dot{\rho}_{11}(t) &= \Gamma_1 \rho_{00}(t) - \Gamma_1 \rho_{11}(t) \\
\dot{\rho}_{01}(t) &= -\frac{\Gamma_1 + \Gamma_1}{2} \rho_{01}(t),
\end{align*}
\]

where we have defined the respective excitation and relaxation rates (see Eq. (4.32))

\[
\begin{align*}
\Gamma_1 &= \gamma (\bar{n}(\omega_{qb}, T) + 1) \\
\Gamma_\uparrow &= \gamma \bar{n}(\omega_{qb}, T).
\end{align*}
\]

The solutions for the diagonal elements are given by

\[
\begin{align*}
\rho_{00}(t) &= \frac{(\Gamma_1 + e^{-((\Gamma_1 + \Gamma_1)\gamma)}\Gamma_\uparrow)\rho_{00}(0)}{\Gamma_1 + \Gamma_1} + \frac{\Gamma_1(1 - e^{-(\Gamma_1 + \Gamma_1)\gamma})\rho_{11}(0)}{\Gamma_1 + \Gamma_1} \\
\rho_{11}(t) &= \frac{\Gamma_1(1 - e^{-(\Gamma_1 + \Gamma_1)\gamma})\rho_{00}(0)}{\Gamma_1 + \Gamma_1} + \frac{(\Gamma_1 + e^{-((\Gamma_1 + \Gamma_1)\gamma)}\Gamma_\uparrow)\rho_{11}(0)}{\Gamma_1 + \Gamma_1},
\end{align*}
\]

which describe an exponential decay of $\rho_{00}$ and $\rho_{11}$ towards the stationary values

\[
\rho_{00}^{ss} = \frac{\Gamma_1}{\Gamma_1 + \Gamma_1}, \quad \rho_{11}^{ss} = \frac{\Gamma_1}{\Gamma_1 + \Gamma_1},
\]

which are set by the temperature of the bath. We see that the transversal coupling causes the occupation of the ground and excited state to change
which correspond to an energy exchange with the bath. At zero temperature, \( \tilde{n}(\omega_{qb}, 0) = 0 \) with the corresponding steady state values \( \rho_{00}^s = 1 \) and \( \rho_{11}^s = 0 \), simply stating that for \( T = 0 \) the bath cannot excite the system. The time-scale for the change in occupation probabilities is set by the rates that appear in the exponents of Eq. (4.37)

\[
\frac{1}{T_1} = \Gamma_\downarrow + \Gamma_\uparrow = 2\pi g(\omega_{qb})\kappa(\omega_{qb})^2 \coth \left( \frac{\hbar \omega_{qb}}{2k_B T} \right),
\]

and is called the \( T_1 \)-time. Thus, relaxation is sometimes also referred to as a \( T_1 \)-process. The off-diagonal elements that describe the phase coherence between the basis states \( |0\rangle \) and \( |1\rangle \) decay exponentially with the rate given by \( 1/2T_1 \) (see last line in Eq. (4.35)).

### 4.3.2 Dephasing

Dephasing is a process which arises when the qubit operator coupling to the bath is given by \( \sigma_z \). This is also known as longitudinal coupling since it is parallel to the quantization axis of the qubit. In this case, the master equation in the interaction picture becomes

\[
\dot{\rho} = \frac{\Gamma_\varphi}{2}(\sigma_z \rho \sigma_z - \rho),
\]

where \( \Gamma_\varphi = 4\Re[\Gamma(0)] \) is the dephasing rate. The solution to this equation is given by

\[
\rho(t) = \begin{bmatrix}
\rho_{00} & e^{-\Gamma_\varphi t} \rho_{01} \\
e^{-\Gamma_\varphi t} \rho_{10} & \rho_{11}
\end{bmatrix}.
\]

As opposed to relaxation, the longitudinal coupling does not induce any energy exchange between the system and bath. The occupation probabilities are therefore not affected by dephasing. However, we see that the coherence between the states \( |0\rangle \) and \( |1\rangle \) decay exponentially with the rate \( \Gamma_\varphi \). For \( t \to \infty \) the initial pure state is transformed into a classical mixture with no information about the relative phase between \( |0\rangle \) and \( |1\rangle \).

### 4.4 Errors as quantum operations

We note that the density matrix subject to relaxation or dephasing as given in Eqs. (4.37) and (4.41) can be written on the form

\[
\rho(t) = \sum_i E_i \rho E_i^\dagger,
\]
with the operators $E_i$ depending on the particular error process. For relaxation they are given by

$$E_0 = \sqrt{p} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - \gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \sqrt{p} \begin{bmatrix} 0 & 0 & \sqrt{\gamma} \\ 0 & 0 & 0 \\ \sqrt{1 - \gamma} & 0 & 1 \end{bmatrix},$$

$$E_2 = \sqrt{1 - p} \begin{bmatrix} \sqrt{1 - \gamma} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \sqrt{1 - p} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma} \\ 0 & \sqrt{1 - \gamma} & 0 \end{bmatrix},$$

(4.43)

with the parameters $\gamma$ and $p$ defined as

$$p = \frac{\Gamma_i}{\Gamma_i + \Gamma_i}, \quad \gamma(t) = 1 - e^{-t/T_1}.$$ 

(4.44)

Similarly, Eq. (4.41) can be written as

$$\rho(t) = p(t)\rho + (1 - p(t))\sigma_z \rho \sigma_z, \quad p(t) = \frac{1}{2} (1 + e^{-\Gamma_i t}).$$

(4.45)

The evolution of the density matrix given in Eq. (4.42), together with the operators in Eqs. (4.43) and (4.45), constitute two examples of the so-called operator sum representation of a broad class of transformations of the density matrix, called operations. An operation, $\Phi[\rho]$, is a completely positive, convex linear mapping which maps positive operators to positive operators, see e.g. [65]. If this mapping furthermore satisfies $0 \leq \text{tr } \Phi[\rho] \leq 1$ there is a theorem due to Kraus [66] stating that $\Phi[\rho]$ can be represented as

$$\Phi[\rho] = \sum_i E_i \rho E_i^\dagger,$$

(4.46)

where the operators $E_i$ satisfy $\sum_i E_i^\dagger E_i \leq 1$. The operator sum representation has a very straightforward interpretation in the context of indirect measurements which will be discussed in Sec. 5.2. Eq. (4.45) allows us to interpret dephasing as a process which flips the phase of the qubit density matrix with probability $1 - p(t)$ and leaves the qubit unaffected with probability $p(t)$. This type of error is thus sometimes referred to as a phase-flip process and the representation in Eq. (4.45) is extensively used when investigating the effect of the phase-flip error-correction code in Paper IV.

4.5 The master equation with a time-dependent Hamiltonian

In Paper IV, we consider a qubit coupled to a tunable cavity where the frequency of the cavity is tuned in and out of resonance with the qubit
level splitting to generate a qubit-qubit interaction. The two regimes clearly correspond to different dynamics where the Hamiltonian is given by

\[
H_S = \begin{cases} 
H_0 = \omega_c (a^+ a + \frac{1}{2}) - \frac{\omega_{qb}}{2} \sigma_z, & \text{off resonance} \\
H_{JC} = H_0 + i \frac{g}{2} (a^+ \sigma_- - a \sigma_+), & \text{on resonance}
\end{cases}
\] (4.47)

for the off-resonant and resonant periods respectively. It is clear that the states used to write the operator \( A(t) \) in Eq. (4.15), as well as the transition frequencies \( \omega_{mn} \) will depend on the energy eigenbasis and thus be different depending on which of the Hamiltonians in Eq. (4.47) that is used. The dynamics in the different regimes will thus be governed by different master equations. In this section, we consider how to treat this case and what approximations must be done to derive a simple master equation. In Paper IV, we consider a longitudinal coupling to the environment where the system operators in Eq. (4.12) are given by \( \sigma_z \) and \( n = a^+ a \) for the qubit and cavity. The eigenstates and corresponding eigenenergies of \( H_S \) in the off-resonant case are given by

\[
|n; g/e\rangle, \quad \omega_{n;g/e} = \omega_c (n + \frac{1}{2}) \mp \frac{\omega_{qb}}{2}.
\] (4.48)

where the upper and lower sign correspond to the respective ground and excited state of the qubit. The situation is different in the resonant case, where the eigenstates and energies are those of the Jaynes-Cummings Hamiltonian [67]

\[
|n; \pm\rangle = \frac{1}{\sqrt{2}}(|g; n\rangle \pm i|e; n-1\rangle), \quad \omega_{n;\pm} = \omega n \mp \frac{g \sqrt{n}}{2}, \\
\omega_{g;0} = 0.
\] (4.49)

Given the system operators \( \sigma_z \) and \( n \), it is clear that, in the off-resonant case, all transition frequencies \( \omega_{n,g/e; n,g/e} \) are zero and the coupling to the bath is given by \( 2R[\Gamma(0)] \). In the resonant case however, the matrix elements in Eq. (4.15) become

\[
\begin{align*}
\langle m; \pm | \sigma_z | n; \pm \rangle &= 0 \\
\langle m; \pm | \sigma_z | n; \mp \rangle &= \delta_{mn} \\
\langle m; \pm | n; \mp \rangle &= \left( n - \frac{1}{2} \right) \delta_{mn} \\
\langle m; \pm | n; \mp \rangle &= \frac{1}{2} \delta_{mn}.
\end{align*}
\] (4.50)

allowing for transitions between states that differ in energy by \( \omega_{n;\pm,n;\mp} = \pm g \sqrt{n} \) and \( \omega_{n;\pm,n;\pm} = 0 \). Different terms in the master equation will thus have different rates. In the case when the bath has no structure on the energy
scale $g\sqrt{n}$, the factor $\pi g(|\omega|)\kappa(|\omega|)^2$ in $\mathcal{R} [\Gamma(\omega)]$ can however be considered constant. Moreover, if we consider bath temperatures such that $k_B T \gg \hbar g$, the Bose occupation number can be approximated by $\bar{n}(g\sqrt{n}, T) \approx \bar{n}(0, T)$. In this case we may safely write $\mathcal{R} [\Gamma(\pm g\sqrt{n})] \approx \mathcal{R} [\Gamma(0)]$, leaving a single rate in the master equation. Since the rate is now a constant for all the transitions, it can be taken out of the sums in Eq. (4.16), which is then transformed back to the Schrödinger picture. This gives the master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_S, \rho] + 2\mathcal{R}[\Gamma_c(0)] \left( n\rho n - \frac{1}{2}\{n^2, \rho\} \right) + 2\mathcal{R}[\Gamma_{qb}(0)](\sigma_z \rho \sigma_z - \rho), \quad (4.51)$$

for both the resonant and off-resonant situation where we have indicated that the qubit and cavity generally couple with different strengths to the baths. We emphasize that the derivation above can only be done when the rates for different transitions are approximately equal.

### 4.6 The transmon master equation

In Paper V, we simulate the evolution of the state of a transmon qubit including the effects of relaxation and dephasing. The master equation governing the dynamics of the system density matrix is given by

$$\dot{\rho} = -i[H_D, \rho] + \sum_{j=0}^{\infty} \gamma_{1,j} \mathcal{D}[|j\rangle \langle j+1|] \rho + \sum_{j<i} \frac{\gamma_{\phi,ij}}{2} \mathcal{D}[|i\rangle \langle j| - |j\rangle \langle i|] \rho, \quad (4.52)$$

where $\gamma_{1,j}$ is the relaxation rate for the transition between the energy eigenstates $|j+1\rangle$ and $|j\rangle$. Here we assume that the relaxation is induced through coupling of charge on the superconducting island to external degrees of freedom. In this case, the coupling strengths are proportional to the matrix element $\langle j| n |j+1\rangle \sim \sqrt{j+1}$, where $n$ is the charge operator on the island [26]. This leads to the scaling $\gamma_{1,j} = \gamma_1(j+1)$ of the relaxation rates which are normalized to the coupling energy of the first transition so that $\gamma_1$ becomes the usual relaxation rate within the qubit subspace. The form of the dephasing super-operator is chosen such that the rates $\gamma_{\phi,ij}$ can be interpreted as the rate for which phase coherence is lost between the states $|i\rangle$ and $|j\rangle$, given that the other rates are set to zero. Since detailed studies of the origins of the relaxation and dephasing processes in the transmon system are ongoing, the master equation in Eq. (4.52) has not been justified from a microscopic point of view. However, the super-operators are plausible from physical arguments and the results derived with the above model agrees with
experiment, and thus seems to correctly describe the dynamics of the open system [30]. The Hamiltonian $H_D$ is given in Eq. (3.46) and includes the time-dependent drive used to realize the single qubit gates as described in Sec. 3.4.
Chapter 5

Measurements in quantum mechanics

In this chapter we review the theory of quantum measurements. Sec. 5.1 gives the standard formalism of projective measurements which in Sec. 5.2 is extended to the theory of indirect measurements. With this background we show how non-projective measurements appear in circuit QED. In Sec. 5.4 we derive two stochastic master equations, applicable to qubit measurements, from a microscopic model and discuss the various approximations made.

5.1 Projective measurements

In quantum mechanics, any measurable property of a system can be represented by an eigenvalue \( \lambda_i \) of some hermitian operator \( O \)

\[
O = \sum_i \lambda_i |i\rangle \langle i| ,
\]

(5.1)

where \( O |i\rangle = \lambda_i |i\rangle \). Given the measurement outcome \( \lambda_i \) the von Neumann projection postulate [68] states that the post-measurement state is given by

\[
\rho_i = \frac{1}{p_i} \Pi_i \rho \Pi_i^\dagger,
\]

(5.2)
where \( p_i \) is the probability for that particular measurement outcome,

\[
p_i = \text{tr}(\Pi_i \rho \Pi_i^\dagger),
\]

and \( \Pi_i \) is the projection operator onto the subspace belonging to \( \lambda_i \) satisfying

\[
\Pi_j \Pi_i = \delta_{ij} \Pi_i.
\]

Note that, due to the non-unitary nature of the measurement (\( \Pi_i^\dagger \Pi_i \neq 1 \)), the state in Eq. (5.2) must be normalized by hand to conserve probability. We will use Eqs. (5.2) and (5.3) as the starting point for the description of indirect quantum measurements which naturally lead to the formulation of conditional evolution and the corresponding stochastic master equation.

### 5.2 Indirect measurements

In this section we consider the case where the system of interest is allowed to interact with a second system which we will call the probe. After the interaction has taken place the probe is subject to a projective measurement as described in Sec. 5.1. Due to the interaction, the probe and system will generally be entangled and a measurement on the probe will thus inevitably affect the system degrees of freedom. We start by considering, in very general terms, the unitary evolution of the combined system, followed by a strong measurement. To this end, we consider the density matrix \( \rho(t) \) on the combined system-probe Hilbert space \( \mathcal{H}_s \otimes \mathcal{H}_p \) where the subscripts denote system and probe respectively. The unitary time evolution for \( \rho(t) \) is given by

\[
\rho(t) = U(t) \rho(0) U^\dagger(t),
\]

where \( U(t, t_0) \) is the time evolution operator describing the interaction between the system and probe. Following the previous section we now make a projective measurement on the probe, labeling the outcomes of this measurement \( a_i \) and their corresponding probabilities to occur \( p_i = \text{tr}[|i\rangle \langle i| \rho |i\rangle \langle i|] \), where \( |i\rangle \) is the state of the environment corresponding to \( a_i \). For a given measurement result the state of the system \( \rho^s \) after the measurement can be obtained by tracing out the probe degrees of freedom

\[
\rho^s_i(t) = \frac{1}{p_i} \langle i| U(t) (\rho^s(0) \otimes |\psi_p\rangle \langle \psi_p|) U^\dagger(t) |i\rangle = \frac{1}{p_i} \Omega_i(t) \rho^s(0) \Omega_i^\dagger(t),
\]

where we assume that the initial state of the combined system was given by the product state \( \rho(0) = \rho^s(0) \otimes |\psi_p\rangle \langle \psi_p| \) and the probe state was pure. The
operator $\Omega_i$ is defined as
\[ \Omega_i(t) = \langle i | U(t) | \psi_p \rangle. \] (5.7)

Using Eq. (5.7) the probabilities $p_i$ are given solely in terms of the system degrees of freedom
\[ p_i(t) = \text{tr}[\Omega_i(t) \rho_s \Omega_i^\dagger(t)] \equiv \text{tr}[E_i \rho_s], \] (5.8)

where we have used the cyclic property of the trace operation and defined the operator $E_i = \Omega_i^\dagger \Omega_i$ which is called the effect of the measurement outcome $a_i$ [66]. Given the effects, the probabilities for all measurement outcomes can be calculated if we know the state of the system. It is important to note that, as opposed to the projection operators, the $\Omega_i$’s are not necessarily orthogonal $\Omega_i \Omega_j \neq \delta_{ij} \Omega_i$. However, they obey the normalization condition
\[ \sum_i \Omega_i^\dagger \Omega_i = 1, \] (5.9)

as can be seen from the definition in Eq. (5.7), which guarantees conservation of probability $\sum p_i = 1$. We note that the indirect measurement maps the initial density matrix of the system to the one given in Eq. (5.6). We recognize this to be on the form given by the Kraus representation theorem which was discussed in Sec. 4.4, and we see that this type of evolution arises naturally when the system of interest is allowed to interact with a probe that is subject to measurement. From this, it is clear that the errors considered in Sec. 4.4 can be interpreted as non-selective measurements, where no record is kept of the measurement result. This means that the states corresponding to different measurement results must be re-mixed, giving the mixed state density matrices of Eqs. (4.42) and (4.45).

So far everything has been quite general and we have not specified in detail the operators $\Omega_i$, which will in general depend on the interaction $U(t)$ and the measurement basis $|i\rangle$. In Sec. 5.4 we treat two explicit cases, where the operators $\Omega_i$ are derived from a microscopic model, and show how this leads a stochastic master equation.

### 5.3 Measuring entanglement in cQED

As an example of a non-projective measurement, we consider the situation in Paper VI where the value of the Mermin operator [38]
\[ M = \sigma_1^x \sigma_2^x \sigma_3^x + \sigma_1^y \sigma_2^y \sigma_3^y + \sigma_1^x \sigma_2^x \sigma_3^y + \sigma_1^y \sigma_2^y \sigma_3^x, \] (5.10)
is extracted via a parity detection in circuit QED. As explained in Paper VI, such a measurement accesses the operator $\sigma_1^z + \sigma_2^z + \sigma_3^z$ from which the parity $\sigma_1^z \sigma_2^z \sigma_3^z$ can be easily inferred. Using single-qubit rotations, one can access the different operators appearing in Eq. (5.10). Since each operator may take on two values $\pm 1$, there are two effects $E_{\pm 1}$ corresponding to the two different measurement outcomes. In the case of an imperfect measurement these effects will generally deviate from that of a simple projective one. In the case when the effects are diagonal in the measurement basis we write these in the form of a weighted projector sum

$$E_{\pm 1} = \sum_{k=\pm 1} w_{k,\pm 1} \Pi_k,$$  

(5.11)

where the weights $w_{k,\pm 1}$ can be interpreted as the conditional probabilities for the outcome $\pm 1$, given that the system was in the subspace $k$. Thus the weights $w_{\pm 1,\pm 1}$ describe the tendency for the measurement to assign the outcome $\pm 1$ with the correct subspace while $w_{\mp 1,\pm 1}$ give the probabilities for assigning $\pm 1$ to the wrong subspace. From this it is clear that all $w_{k,\pm 1}$ are classically well-defined probabilities which can be used to define a classical measurement scheme. In principle, the effects can be determined through detector tomography [69] and are not necessarily diagonal in the measurement basis. In this case, the effects must first be diagonalized to be amenable to the above interpretation. If we restrict the analysis to measurements with only two outcomes, the normalization condition in Eq. (5.9) guarantees that the two effects can be diagonalized simultaneously. To see this we consider $E_{+1}$ to be diagonal. Eq. (5.9) then implies that $E_{-1} = 1 - E_{+1}$ is diagonal in the same basis. Thus $E_{+1}$ and $E_{-1}$ can be diagonalized simultaneously and we write

$$E_{\pm 1} = \sum_{k=\pm 1} w_{k,\pm 1} \tilde{\Pi}_k.$$  

(5.12)

It is important to realize that in this case $\tilde{w}_{k,\pm 1}$ labels the conditional probabilities with respect to the eigenstates of $E_{\pm 1}$ which generally do not coincide with the eigenstates of the measured operator. Any local hidden variable theory however, will be stated in terms of probabilities to measure such states, since these correspond to well-defined classical measurement results. Because of this, it is clear that we run into problems if $\tilde{w}_{k,\pm 1}$ is to be used in the context of a LHV theory. Luckily it is possible to construct a dispersive measurement with diagonal effects only. This will be described in the remaining
part of this section. Note that the parity of $\sigma_z^1\sigma_z^2\sigma_z^3$ can be written

$$\langle \psi | \sigma_z^1\sigma_z^2\sigma_z^3 | \psi \rangle = \sum_{i,j,k=0}^{1} (-1)^{i+j+k}p_{ijk}$$

for any state $|\psi\rangle$ where $p_{ijk}$ is the probability to detect the system in the state $|ijk\rangle$, where $i, j, k = 0, 1$. The crucial point now lies in the fact that for sufficiently low temperatures there will be no leaking of states into the $|111\rangle$-subspace associated with the largest signal $\langle \sigma_z^1 + \sigma_z^2 + \sigma_z^3 \rangle = 3$. Thus we may construct a measurement $B$ by assigning the outcomes “0” and “1” to the respective cases when the signal is smaller or larger than a preset threshold $\nu$. By setting $\nu$ sufficiently high, the probability for false positive events with respect to the outcome “1” is suppressed and the corresponding effect reads

$$E_1 = \alpha |111\rangle \langle 111|,$$

which is diagonal in the measurement basis. Clearly $\alpha$ is the probability to detect “1” given that the states was in the $|111\rangle$-subspace. The remaining probabilities in Eq. (5.13) can be determined analogously by swapping the states of one, two or three qubits using single qubit rotations. The value of $\langle \sigma_z^1\sigma_z^2\sigma_z^3 \rangle$ in Eq. (5.13) is thus given by

$$\langle \sigma_z^1\sigma_z^2\sigma_z^3 \rangle = \alpha \langle \sigma_z^1\sigma_z^2\sigma_z^3 \rangle_{\text{ideal}}$$

where $\langle \sigma_z^1\sigma_z^2\sigma_z^3 \rangle_{\text{ideal}}$ is the value in the absence of measurement imperfections ($\alpha = 1$). The above scheme, together with single qubit rotations, can be used to measure all the parity operators appearing in Eq. (5.10). The measured value of the Mermin operator for a GHZ-state is thus given by

$$\langle \text{GHZ} | M | \text{GHZ} \rangle = \alpha \langle \text{GHZ} | M | \text{GHZ} \rangle_{\text{ideal}} = 4\alpha.$$

Any LHV theory, on the other hand, puts bounds on $\langle M \rangle$ such that $-2 \leq \langle M \rangle \leq 2$. If $\alpha \geq 1/2$ this inequality can be violated, thereby proving the existence of non-classical correlations in our system. 1.

1As discussed in Paper VI, we accept that a loop-hole free violation of LHV theories cannot be made in current circuit QED systems due to the time-like separation of the qubits.
5.4 The stochastic master equation

In this section we consider the case when the interaction between the probe and the system introduced in Sec. 5.2 takes place during a very short time $\tau = t - t_0 \to 0$. In this case the unitary evolution and measurement can be combined into a dynamical equation,

$$\rho_i(t + dt) = \frac{1}{p_i(dt)} \Omega_i(dt) \rho(t) \Omega_i^\dagger(dt),$$

(5.17)

for the density matrix, conditioned on the outcome of the measurement. The subscript $i$ labels the measurement outcome. Due to the probabilistic nature of quantum measurement theory, any equation of this form will describe stochastic evolution and is thus suitably called a *Stochastic master equation* (SME) or a quantum trajectory equation. Such an equation can be used to study the evolution of a system conditioned on a continuous measurement as was done in Paper VI. In the following sections we derive two such SME’s corresponding to two different measurements: direct photodetection and homodyne detection. This follows closely the derivation given in Ref. [65].

5.4.1 Direct photodetection

We consider a two level system (qubit) coupled to a continuum of electromagnetic modes. Our aim is to describe the conditional evolution of the qubit density matrix when the excitations in the modes are measured through direct photo-detection. The Hamiltonian for the combined system reads

$$H = H_{qb} + H_{bath} + H_{int},$$

(5.18)

where

$$H_{qb} = -\frac{\omega_0}{2} \sigma_z$$

$$H_{osc} = \sum_i \omega_i b_i^\dagger b_i$$

$$V = \sigma_x \sum_i \kappa_i^* b_i + \kappa_i b_i^\dagger + V_{qb},$$

(5.19)

where we have set $\hbar = 1$ and $\omega_0$ and $\omega_i$ are the resonance frequencies of the qubit and mode $i$ respectively, $\kappa_i$ is the coupling between the qubit and the
i\textsuperscript{th} mode of the bath and $V_{qb}$ may contain additional qubit terms only. In the interaction picture of $H_{qb} + H_{osc}$, the Hamiltonian reads

$$V_I(t) = e^{i\omega_0 t} \sigma_+ B(t) + e^{-i\omega_0 t} \sigma_- B^\dagger(t) + V_{qb}^I(t)$$

(5.20)

where fast oscillating terms have been neglected in the rotating wave approximation and the superscript $I$ denotes the interaction picture. The shorthand

$$B(t) = \sum_k \kappa_j b_j e^{-i\omega_j t},$$

(5.21)

has been introduced for convenience. Considering a short interaction time, using time-dependent perturbation theory, the time evolution operator is given by

$$U(t,t_0) \approx 1 - i \int_{t_0}^t dt' V_I(t') - \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V_I(t') V(t'').$$

(5.22)

So far we have not introduced the measurement. This is done by projecting Eq. (5.22) on an appropriate choice of measurement basis. In the case of direct photo-detection, photons emitted by the qubit are directly detected in a photocounter. We can thus distinguish between the events: detecting a photon in mode $j$ or not detecting a photon at all corresponding to a projection onto the Fock states $|1_j\rangle = b_j^\dagger |0\rangle$ and $|0\rangle$. The initial state of the probe is taken to be the vacuum. This gives the operators in Eq. (5.7)

$$\begin{align*}
\Omega_0(t,t_0) &= \langle 0 | U(t,t_0) | 0 \rangle \\
\Omega_{1j}(t,t_0) &= \langle 1_j | U(t,t_0) | 0 \rangle, \quad \forall j.
\end{align*}$$

(5.23)

Note that we are explicitly assuming that the photodetector only detects zero or a single photon in the time interval $[t,t_0]$. This is perfectly reasonable since we are considering the short time limit were the probability of a two-photon detection is negligible. Starting with $\Omega_0$ we get

$$\begin{align*}
\Omega_0(t,t_0) &= \langle 0 | 1 - i \int_{t_0}^t dt' V_I(t') - \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V_I(t') V_I(t'') | 0 \rangle.
\end{align*}$$

(5.24)

The second term only contains unpaired creation and annihilation operators and is thus given by

$$i \int_{t_0}^t dt' \langle 0 | V_I(t') | 0 \rangle = i\tau V_{qb}^I,$$

(5.25)
where \( \tau = t - t_0 \). The third term gets a contribution from the \( B(t')B(t'') \)-term yielding

\[
\int_{t_0}^{t} dt' \int_{t_0}^{t''} dt'' \langle 0 | V_j(t') V_j(t'') | 0 \rangle = \int_{t_0}^{t} dt' \int_{t_0}^{t''} dt'' \sum_j |\kappa_j|^2 e^{i(\omega_j - \omega)(t' - t'')} \sigma_+ \sigma_-
\]

\[
\equiv g_0 \sigma_+ \sigma_- , \tag{5.26}
\]

where \( g_0 \) is to be calculated later and we have neglected terms \( \propto \tau^2 \). The operator corresponding to not detecting a photon is thus given by

\[
\Omega_0(t, t_0) = 1 - i \tau V_{qb}^I - g_0 \sigma_+ \sigma_- . \tag{5.27}
\]

We now turn to the operator describing the detection of a photon in mode \( j \). In this case it is clear that the constant term in Eq. (5.22) vanishes. This is also the case for the third term in Eq. (5.22) since the field operators either leaves the photon number invariant or changes it by two. This leaves us with

\[
\Omega_{1j}(t, t_0) = -i \int_{t_0}^{t} dt' \langle 1_j | V_j(t') | 0 \rangle = -i \kappa_j^* \int_{t_0}^{t} dt' e^{-i(\omega_0 - \omega_j)t'} \sigma_- \equiv f_j \sigma_- . \tag{5.28}
\]

In general each operator \( \Omega_{1j} \) would correspond to a different measurement result \( j \) and thus a different evolution of the qubit density matrix. In the photodetection case however, there is no distinction between a detection in mode \( j \) and \( j' \). Thus, we can view this as a non-selective measurement where the information about the measurement outcome is thrown away and the different states corresponding to the different outcomes are remixed, leaving us with the unnormalized state after the detection of a photon

\[
\hat{\rho}_1(t) = \sum_j \Omega_{1j}(t, t_0) \rho(t_0) \Omega_{1j}(t, t_0)^\dagger = \sum_j |f_j|^2 \sigma_- \rho(t_0) \sigma_+ . \tag{5.29}
\]

We are now in a position to write down the qubit state, properly normalized, conditioned on the detection of zero or one photon

\[
\rho_0(t) = \frac{(1 - i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-) \rho(t_0)(1 + i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-)}{\langle (1 - i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-)(1 + i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-) \rangle},
\]

\[
\rho_1(t) = \frac{\sigma_- \rho(t_0) \sigma_+}{\langle \sigma_+ \sigma_- \rangle} , \tag{5.30}
\]

where \( \langle ... \rangle = \text{tr}[... \rho(t_0)] \). The corresponding probabilities for the measurement outcomes

\[
p_0 = \langle (1 - i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-)(1 + i \tau V_{qb}^I - g_0 \sigma_+ \sigma_-) \rangle
\]

\[
p_1 = \sum_j |f_j|^2 \langle \sigma_+ \sigma_- \rangle . \tag{5.31}
\]
We now proceed to calculate \( g_0 \) and \( \sum_j |f_j|^2 \) and show that they are \( \propto \tau \).

This is equivalent to Fermi’s golden rule where the decay is quadratic in the coupling to the bath. For our purposes we assume the bath modes to be well approximated by a continuum in frequency space with no structure on the scale \( 1/\tau \) (see Appendix A). We find that

\[
\sum_j |f_j|^2 = 2g_0 = 2\pi|\kappa(\omega_0)|^2 D(\omega_0) \tau \equiv \gamma_0 \tau, \tag{5.32}
\]

where \( D(\omega) \) is the bath density of states and we have ignored the Lamb shift.

Eq. (5.30) and Eq. (5.31) can now be expanded to first order in \( \tau \) yielding the following two states conditioned on the detection of zero or one photons respectively

\[
\rho_0(t) = \rho(t_0) - i[V_q^T, \rho(t_0)] \tau - \frac{\gamma_0 \tau}{2} \{\sigma_+, \rho(t_0)\} + \gamma_0 \langle \sigma_+ \sigma_- \rangle \tau \rho(t_0),
\]

\[
\rho_1(t) = \frac{\sigma_- \rho(t_0) \sigma_+}{\langle \sigma_+ \sigma_- \rangle}, \tag{5.33}
\]

where we have defined the anti-commutator \( \{A, B\} = AB + BA \). The probabilities are given by

\[
p_0 = 1 - \gamma_0 \tau \langle \sigma_+ \sigma_- \rangle,
\]

\[
p_1 = \gamma_0 \tau \langle \sigma_+ \sigma_- \rangle, \tag{5.34}
\]

obviously satisfying \( p_0 + p_1 = 1 \). Equivalently, we may write the change of the state during the time interval \( \tau \rightarrow dt \) as

\[
d\rho_0 = -i[V_q^T, \rho] dt - \frac{\gamma_0 \tau}{2} \{\sigma_+, \rho\} dt + \gamma_0 \langle \sigma_+ \sigma_- \rangle \rho dt, \tag{5.35}
\]

\[
d\rho_1 = \frac{\sigma_- \rho \sigma_+}{\langle \sigma_+ \sigma_- \rangle} - \rho. \tag{5.36}
\]

The above set of equations describes the evolution of the qubit state conditioned on the detection of photons with the measurement statistics given by Eqs. (5.34). However, it turns out that there is an elegant way to describe the conditional evolution in terms of a Poisson process. To this end we define the measurement record \( N(t) \) to be the number of detected photons as a function of time. As discussed above we consider the case when the number of detected photons in the time interval, \( dt \) can take values zero or one. We can thus define the increment \( dN(t) = N(t + dt) - N(t) \) with the following properties

\[
dN^2(t) = dN(t),
\]

\[
E[dN(t)] = \gamma_0 \langle \sigma_+ \sigma_- \rangle dt. \tag{5.37}
\]
These are the defining properties of a Poisson process where the probability of detecting $n$ photons in the time interval $[0, t]$ is given by

$$P(N(t) = n) = \frac{\Gamma(t)^n}{n!} e^{-\Gamma(t)},$$  \hspace{1cm} (5.38)

where $\Gamma(t)$ is the average number of photons detected in the time interval $[0, t]$

$$\Gamma(t) = \int_0^t \gamma_0 \langle \sigma_+ \sigma_- \rangle \, dt'.$$  \hspace{1cm} (5.39)

We note that the decay rate is a time dependent quantity due to the dynamics of the qubit state. As expected $\Gamma(t)$ is maximized/minimized for the excited/ground state of the qubit. We see that the with probability $1 - O(dt)$ the measurement yields a null result that is $dN(t) = 0$, with the corresponding smooth evolution of $\rho(t)$ given by Eq. (5.35). Occasionally, with probability $\propto O(dt)$, $dN(t) = 1$ and the system experiences a quantum jump (Eq. (5.36)), conditioned on the detection of a photon. The SME for direct photo-detection can thus be written as

$$d\rho = \left( \frac{\sigma_- \rho \sigma_+}{\langle \sigma_+ \sigma_- \rangle} - \rho \right) dN(t) - \left( i[V_{qb}^I, \rho] + \frac{\gamma_0}{2} \{\sigma_+ \sigma_-, \rho\} - \gamma_0 \langle \sigma_+ \sigma_- \rangle \rho \right) dt.$$  \hspace{1cm} (5.40)

### 5.4.2 Homodyne detection

We now turn to a second example of a quantum trajectory equation, where the light coming from the qubit is mixed with a strong coherent light coming from a local oscillator using a 50 : 50 beam splitter and detected with two detectors $D_1$ and $D_2$, see Fig 5.1. As before we write down the Hamiltonian for the combined system in the interaction picture

$$V_I(t) = e^{i\omega_0 t} \sigma_+ A(t) + e^{-i\omega_0 t} \sigma_- A^\dagger(t) + e^{i\omega_0 t} \beta^* B(t) + e^{-i\omega_0 t} \beta B^\dagger(t) + V_{qb}^I,$$  \hspace{1cm} (5.41)

with $A(t)$ defined as in Eq. (5.21) and $B(t)$ defined analogously. The presence of the strong coherent field is taken into account through a coherent driving of the $b_j$-modes coupling to the local oscillator. The homodyne measurement is similar to the direct photo-detection in the sense that the detectors still detect photons. Hence, the Fock states are used as the basis for the projective measurement of the field. However, since more detectors are involved, the previous basis has to be expanded to include all modes depicted in Fig. 5.1. In analogy with Sec. 5.4.1 the measurement basis is chosen to be the vacuum $|0\rangle$ (no detected photons) and the one photon states $d^{\dagger}_{1j} |0\rangle$ (a photon
Figure 5.1: A schematic view of a homodyne measurement. Using a 50:50 beamsplitter, the light from the qubit is mixed with a strong, coherent signal coming from the local oscillator. The number of photons are then detected in the two detectors $D_1$ and $D_2$ and subtracted yielding the photocurrent $j(t)$. (See text for details).

in $D_1$) and $d^\dagger_{2j} |0\rangle$ (a photon in $D_2$) of the modes that couple to the detectors. Another complication comes from the fact that the Hamiltonian, and thus $U(t,t_0)$ is written in terms of $a_j$ and $b_j$. We must thus relate these to the modes that couple to the detectors. Given a 50:50 beamsplitter and including a phase shift of $\pi/2$ this relation is given by [67]

\[
\begin{align*}
    d_{1j} &= \frac{1}{\sqrt{2}} (a_j + ib_j) \\
    d_{2j} &= \frac{1}{\sqrt{2}} (a_j - ib_j).
\end{align*}
\] (5.42)

As for the case of direct photo detection we proceed to calculate the operators that correspond to the measurement of zero and one photon in either detector. Projecting on the vacuum state $|0\rangle$ gives

\[
\Omega_0(t,t_0) = |0\rangle \langle 0| - i \int_{t_0}^{t} dt' V_{qb}^I(t') - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_{qb}^I(t') V_{qb}^I(t'') |0\rangle,
\] (5.43)

where the first and second terms are as in Eq. (5.25) and the contribution to the third now comes from the combinations $a_j a_j^\dagger = b_j b_j^\dagger = (c_j c_j^\dagger + d_j d_j^\dagger)/2$
+ terms that do not contribute to the matrix element. By direct comparison with Eq. (5.27) we can write down the operator corresponding to the detection of zero photons as

$$\Omega_0(t, t_0) = 1 - i\tau V_{q\beta}^I - g_0(\sigma_+\sigma_- + |\beta|^2). \tag{5.44}$$

We now turn to the operator for the detection of one photon in \(D_1\) or \(D_2\). As before, the only contribution is from the term in the time evolution operator containing a single creation operator

$$\Omega_{1/2,j}(t, t_0) = -i \int_{t_0}^{t} dt' \langle 0 | (a_j \pm ib_j)V_{q\beta}^I(t') | 0 \rangle = f_j \sqrt{2} (\sigma_- \pm i\beta), \tag{5.45}$$

where \(f_j\) is defined as in Eq. (5.28) and we have adopted the convention that the upper and lower sign is for detector \(D_1\) and \(D_2\) respectively. As discussed in Sec. 5.4.1 the detectors do not distinguish between photons in the different \(j\)-modes which again allows for a re-mixing of post measurement states. Since the measurement now involves two detectors there will be three different states conditioned on the different outcomes

$$\rho_0(t) = \rho(t_0) - i[V_{q\beta}, \rho(t_0)]\tau - \frac{g_0}{2} \{\sigma_+\sigma_-, \rho(t_0)\} + \gamma_0(\sigma_+\sigma_-)\tau \rho(t_0),$$

$$\rho_{1/2}(t) = \frac{(\sigma_- \pm i\beta)\rho(t_0)(\sigma_+ \mp i\beta^*)}{\langle(\sigma_+ \pm i\beta^*)(\sigma_- \pm i\beta)\rangle}, \tag{5.46}$$

with the corresponding probabilities

$$p_0 = 1 - \gamma_0\tau(\langle\sigma_+\sigma_-\rangle + |\beta|^2),$$

$$p_{1/2} = \frac{\gamma_0\tau}{2} \langle(\sigma_+ \mp i\beta^*)(\sigma_- \pm i\beta)\rangle. \tag{5.47}$$

Eq. (5.46) and Eq. (5.47) can be used to write down a stochastic equation similar to Eq. (5.40). Since there are two measurement records, we now need to define two Poisson processes \(N_1(t)\) and \(N_2(t)\) and their corresponding increments \(dN_1(t)\) and \(dN_2(t)\) satisfying

$$dN_i(t)dN_j(t) = \delta_{ij} dN_i(t),$$

$$E[dN_{1/2}(t)] = \gamma_0 \langle(\sigma_+ \mp i\beta^*)(\sigma_- \pm i\beta)\rangle dt, \tag{5.48}$$

where the first equation states the fact that two photons cannot simultaneously be detected in \(D_1\) and \(D_2\). In the ideal case of homodyne detection the local oscillator field is strongly excited \(|\beta| \rightarrow \infty\). To see how this effects the conditional evolution of the qubit it is convenient to re-scale the measurement operators

$$\sigma_- \pm i\beta = \pm i|\beta|e^{i\theta}(1 \mp \epsilon C), \tag{5.49}$$
where we have defined \( C = i e^{-iθ}\sigma_- \) and \( ε = 1/|β| \). From Eq. (5.49) it is clear that in the limit \( ε → 0 \) the measurement operator is mostly proportional to the identity operator. This implies that the back-action of a photo-detection is negligible which is to be expected since, in this limit, the proportion of detected photons giving information about the state of the system is very small. Instead of writing the SME for homodyne detection we investigate the \( |β| → ∞ \) limit by considering the Liouville equation for the density matrix probability density [65]

\[
P[\tilde{ρ}, t + τ] = \int dρP[ρ, t] \sum_m \text{tr}(Ω_m^†Ω_mρ)δ \left[ \frac{Ω_mρΩ_m^†}{\text{tr}(Ω^†Ωρ)} - \tilde{ρ} \right], \tag{5.50}
\]

which describes the change of the probability density functional \( P[ρ, t] \). Here the density matrix \( ρ \) is interpreted as a stochastic variable on the space of positive, hermitian matrices with unity trace (see [65] for details). The expression in Eq. (5.50) has a very intuitive meaning in the framework of selective measurements. For each measurement outcome, the \( δ \)-function describes the jump from the initial state \( ρ \) to the state after the measurement. Each jump occurs with a probability given by the pre-factor. In total this gives a transition probability for each \( m \) describing the of change of the probability density for each \( \tilde{ρ} \). Inserting the post-measurement states and the probabilities from Eq. (5.46) and Eq. (5.47) we arrive at

\[
P[\tilde{ρ}, t] = \int dρP[ρ, t_0] \left( T_0[\tilde{ρ}, t|ρ, t_0] + T_1[\tilde{ρ}, t|ρ, t_0] + T_2[\tilde{ρ}, t|ρ, t_0] \right), \tag{5.51}
\]

where the transition probabilities are given by

\[
T_0[\tilde{ρ}, t|ρ, t_0] = \left( 1 - τγ_0(⟨σ_+σ_-⟩ + |β|^2) \right) \times δ \left[ ρ - iτ \left( [V_q, ρ] - \frac{iγ_0}{2} \{σ_+σ_- - ⟨σ_+σ_-⟩, ρ \} \right) - \tilde{ρ} \right], \tag{5.52}
\]

and

\[
T_{1/2}[\tilde{ρ}, t|ρ, t_0] = \frac{1}{2}τγ_0|β|^2(1 ± εC^†)(1 ± εC)δ \left[ \frac{(1 ± εC)ρ(1 ± εC^†)}{(1 ± εC^†)(1 ± εC)} - \tilde{ρ} \right]. \tag{5.53}
\]

As shown in detail in Appendix C, Eq. (5.52) and (5.53) can be expanded to first and second order in \( τ \) and \( ε \) respectively. Integration with respect to \( ρ \) then gives the equation of motion for the probability distribution

\[
\frac{∂P[ρ, t]}{∂t} = -\frac{∂}{∂ρ} \left( K[ρ]P[ρ, t] \right) + \frac{1}{2} \frac{∂^2}{∂ρ^2} \left( γ_0M_z^2[ρ]P[ρ, t] \right), \tag{5.54}
\]
where $\mathcal{K}$ and $\mathcal{M}$ are super-operators given by

\begin{align}
\mathcal{K}[\rho] &= -i[V_{qb}^I, \rho] + \gamma_0 D_{\sigma_-}[\rho] \\
\mathcal{M}_{\sigma_-}[\rho] &= (\sigma_- - \langle \sigma_- \rangle)\rho + \rho (\sigma_+ - \langle \sigma_+ \rangle).
\end{align}

(5.55)

where $D_{\sigma_-}[\rho] = \sigma_- \rho \sigma_+ - \{\sigma_+ \sigma_-, \rho\}/2$. The expression in Eq. (5.54) can be recognized as a Fokker-Planck equation for the probability density $P[\rho, t]$ with a drift and diffusion term. From stochastic calculus we know that the statistics described by this equation is equivalently described by a stochastic process $\rho(t)$ if this is given by an Ito process \[70\]

\[d\rho = \mathcal{K}[\rho]dt + \sqrt{\gamma_0} \mathcal{M}_{\sigma_-}[\rho]dW(t),\]

(5.56)

where $dW(t)$ is a Wiener increment with the statistical properties

\[E[dW(t)] = 0, \quad \text{Var}[dW(t)] = dt.\]

(5.57)

Eq. (5.56) is the SME for homodyne detection which was also derived in e.g. Ref. \[71\].

We now turn to the signal generated by the detectors. As mentioned above the ideal limit of homodyne detection is when the local oscillator amplitude goes to infinity. In this case the rate of photodetections goes to infinity and it is possible to approximate the Poisson process describing single photon counts with a continuous current plus white noise. The homodyne signal $j(t)\tau_W$ in the time window $[t, t + \tau_W]$, is taken to be the difference between the number of detected photons at detector $D_1$ and $D_2$ respectively, normalized to the local oscillator amplitude

\[j(t)\tau_W = \frac{N_1(\tau_W) - N_2(\tau_W)}{|\beta|}.\]

(5.58)

We will return to the definition of $\tau_W$ and its relation to the previously defined measurement time $\tau$ at the end of this section. As discussed above, $N_i(t)$ is a stochastic variable obeying Poissonian statistics

\[P(N_i(\tau_W) = n) = \frac{(\Gamma_i \tau_W)^n}{n!} e^{-\Gamma_i \tau_W},\]

(5.59)

where $\Gamma_i \tau_W$ is the average number of photons detected during time $\tau_W$. For the case of a strong local oscillator the average number of detected photons goes to infinity and $\Gamma \tau_W \gg 1$. In this case the Poisson distribution can be approximated by a normal distribution $N(\mu, \sigma^2)$ with mean $\mu = \Gamma \tau_W$ and
Figure 5.2: A schematic view of the process $dN(t)$ as a function of time. The measurement time $\tau$ is defined so that not more than one event $dN(t) = 1$ can occur. The time-scale $\tau_W$, where it is possible to approximate this process with a Wiener process, is such that the number of events approaches infinity. From this schematic, it is clear that $\tau_W \gg \tau$.

The current in Eq. (5.58) is thus the difference between two normally distributed random variables with corresponding means and variances

\[\begin{align*}
\mu_1 &= \sigma_1^2 = \frac{1}{2} \tau_W \gamma_0 \langle (\sigma_+ - i\beta^\ast)(\sigma_- + i\beta) \rangle, \\
\mu_2 &= \sigma_2^2 = \frac{1}{2} \tau_W \gamma_0 \langle (\sigma_+ + i\beta^\ast)(\sigma_- - i\beta) \rangle,
\end{align*}\]

which is again normally distributed $N(\mu_\Sigma, \sigma_\Sigma^2)$ with

\[\begin{align*}
\mu_\Sigma &= \mu_1 - \mu_2 = i \tau_W \gamma_0 |\beta| \langle \sigma_+ e^{i\theta} - \sigma_- e^{-i\theta} \rangle, \\
\sigma_\Sigma^2 &= \sigma_1^2 + \sigma_2^2 = \tau_W \gamma_0 (\sigma_+ \sigma_- + |\beta|^2) \approx \tau_W \gamma_0 |\beta|^2.
\end{align*}\]  

In the limit $\tau_W \to 0$ we can thus write the photocurrent

\[j(t)dt = i \gamma_0 (\sigma_+ e^{i\theta} - \sigma_- e^{-i\theta}) dt + \sqrt{\gamma_0} dW(t),\]  

where $dW(t)$ is a Wiener increment having normal distribution with mean $E[dW(t)] = 0$ and variance $\text{Var}[dW(t)] = dt |70|$. Note that the limit $\Gamma \tau_W \gg 1$ above only has meaning if $\tau_W \gg \tau$ where $\tau$ was introduced in Sec. 5.4 as the measurement time such that the detection of two photons could be neglected. In Fig. 5.2, this difference is highlighted. Thus it is only on the time scale $\tau_W$ that the stochastic process describing the photo-current can be approximated by a Wiener process.
5.5 Measurement induced dephasing

From Eq. (5.56), it is clear that the measurement induces a non-unitary evolution on the system state, as is represented by the super-operators $\mathcal{D}$ and $\mathcal{M}$ in Eq. (5.56). We recognize $\mathcal{D}$ from Sec. 4.2, where this super-operator was derived from a transversal qubit-bath coupling leading to a $T_1$-process. Since the homodyne measurement involves the same type of coupling to the probe degrees of freedom, the appearance of $\mathcal{D}$ in the SME is not unexpected. As a consequence, direct homodyne measurement on the qubit makes the system relax to its ground state. From the discussion in Sec. 5.2 it is clear that any measurement will somehow cause a change in the system state. The fact that a homodyne measurement involves exchange of energy with the probe is the reason for the appearance of relaxation in the equation of motion. We now show that, even in the absence of energy exchange, the measurement must cause the system to dephase. This is an inevitable consequence of the entanglement between the system and probe and is thus called measurement induced dephasing.

The process of measurement induced dephasing can be understood by considering the simplest measurement of all, that is the one amounting to the choice between two outcomes. We thus consider the evolution of a qubit coupled to a quantum mechanical probe whose quantum state we denote $|\psi\rangle$. Before the interaction between the object and probe, the joint system is in the state

$$\text{(5.63)} \quad (a|0\rangle + b|1\rangle) |\psi\rangle,$$

where $|0\rangle$ and $|1\rangle$ are the basis states of the qubit with complex amplitudes $a$ and $b$. In order for the probe to be an effective carrier of information, the interaction must produce a correlated state of the form [48]

$$\text{(5.64)} \quad |\Psi\rangle = a|0\rangle |\psi_0\rangle + b|1\rangle |\psi_1\rangle,$$

where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the probe states that correspond to the measured $|0\rangle$ and $|1\rangle$ states of the qubit. To describe the evolution of the qubit degrees of freedom alone, the probe is traced out leaving the reduced density matrix of the qubit

$$\rho_{\text{sys}} = \text{tr}_{\text{probe}}[\rho] = \begin{pmatrix} |a|^2 & ab^* \\ ab & |b|^2 \end{pmatrix}.$$  \hspace{1cm} \text{(5.65)}

To be able to distinguish the two measurement outcomes, the probe states must be sufficiently orthogonal. Hence, to be able to extract any information from the measurement we must demand that $\langle \psi_0 | \psi_1 \rangle \rightarrow 0$ during the time
the measurement is carried out. This in turn makes the off-diagonal elements of the reduced density matrix vanish

\[ \rho_{\text{sys}} \rightarrow \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}, \tag{5.66} \]

destroying the information of the relative phase between \(|0\rangle\) and \(|1\rangle\). After the measurement, the reduced density matrix will only provide information about the probability to find the system in either state. The density matrix in Eq. (5.66) clearly represents a mixed state which can be verified by observing that \(\text{tr}(\rho_{\text{sys}}^2) < 1\). Observe that, since the interaction in Eq. (5.64) did not involve any energy exchange between the qubit and probe, the occupation probabilities \(|a|^2\) and \(|b|^2\) are unaffected. The only effect of the measurement is to destroy the phase coherence between the states \(|0\rangle\) and \(|1\rangle\), which cannot be avoided if any information is to be extracted from the measurement.

From the discussion above, it is clear the measurement time and dephasing rate are intimately connected, a connection that we now wish to quantify. To connect with the above discussion and properly define a measurement time, we consider an SME where, as opposed to Eq. (5.56), the qubit does not exchange energy with the probe. Such an SME was derived in Ref. [72] to describe the dispersive measurement in circuit QED. Here, the cavity degrees of freedom are traced out giving an effective equation for the qubit only,

\[ d\rho = -i[H_{\text{eff}}, \rho]dt + \frac{\Gamma_\varphi}{2} D_{\sigma_z}[\rho] dt + \sqrt{\Gamma_m}\mathcal{M}_{\sigma_z}[\rho] \, dW(t), \tag{5.67} \]

where \(\mathcal{M}_{\sigma_z}[\rho] = (\sigma_z - \langle \sigma_z \rangle)\rho + \rho(\sigma_z - \langle \sigma_z \rangle)\) and \(D_{\sigma_z}[\rho] = \sigma_z\rho\sigma_z - \rho\). We recognize \(D_{\sigma_z}[\rho]\) as the dissipator for pure dephasing given in Eq. (4.40). This is quite natural, since there is no energy exchange between the cavity and qubit in the dispersive regime, such that the only effect of the measurement is to dephase the qubit. The Hamiltonian \(H_{\text{eff}} \propto \sigma_z\) is the effective Hamiltonian for the reduced system and we have omitted all other sources of decoherence apart from that induced by the measurement. We see that the eigenstates of \(\sigma_z\ (\sigma_z |0/1\rangle = \pm |0/1\rangle)\) are stationary states of Eq. (5.67) such that the dispersive measurement of the cavity corresponds to a measurement of the \(\sigma_z\)-operator of the qubit. From this it is also clear that, given a measurement result corresponding to either state \(|0\rangle\) or \(|1\rangle\), a subsequent measurement will yield the same result, which is the definition of a quantum non-demolition measurement [73]. Here, \(\Gamma_\varphi\) is the measurement induced dephasing rate and \(\Gamma_m\) is the measurement rate where \(\Gamma_\varphi = \Gamma_m/2\) [72]. The current is given by (see also Eq. (5.62))

\[ j(t)dt = \Gamma_m \langle \sigma_z \rangle dt + \sqrt{\Gamma_m}dW(t). \tag{5.68} \]
To see how $\Gamma_m$ is related to the measurement time, it is instructive to consider how fast the two orthogonal eigenstates of the measurement can be distinguished in the experiment. To this end we define the integrated current

$$s(t) = \int_0^t j(t')dt',$$

and consider its statistical properties. Since $\int dW(t') = W(t) - W(0)$ has normal distribution $N(0, t)$ [70], the integrated current in Eq. (5.69) is a normally distributed stochastic variable with mean $E[s(t)] = \pm \Gamma_m t$ and variance $\text{Var}[s(t)] = \Gamma_m t$, where the $+$ and $-$ sign of the mean correspond to the two different eigenstates $|0\rangle$ and $|1\rangle$). Given this time dependence, the distributions corresponding to the respective eigenstates of the measurement will separate with time as can be seen in Fig. 5.3. The measurement time

![Graph showing probability distributions](image)

**Figure 5.3:** The probability distributions $P_+(s, t)$ (dashed lines) and $P_-(s, t)$ (solid lines) corresponding to the measurement eigenstates $|0\rangle$ and $|1\rangle$. For $t < t_m$ the two distributions overlap and we cannot distinguish the states. At $t = t_m$ the distributions have separated and a measurement is possible. When $t$ grows beyond $t_m$ the measurement can be considered projective.

is defined as the time for which the overlap between the two distributions is sufficiently small such that a measurement result can be assigned to one of the peaks and thus a definite qubit state. Following Ref. [63], we define the measurement time $t_m$ as the time for which the width of the distribution is
equal to the mean. For a normal distribution $N(\pm \Gamma_m t, \Gamma_m t)$, this gives the measurement time $t_m = 1/\Gamma_m$.

### 5.6 Quantum efficiency and quantum limited measurements

The two time-scales involved in a measurement are given by the measurement induced dephasing time $t_\varphi = 1/\Gamma_\varphi$ and the measurement time $t_m = 1/\Gamma_m$. They provide information about how fast the off-diagonal elements in the density matrix vanish, and how fast the probability distributions for the measurement eigenstates $|0\rangle$ and $|1\rangle$ separate. In a perfect measurement as described by Eq. (5.67), all the information about the state of the system is accounted for and no additional sources of error is present. In this case, these rates are related to each other according to $\Gamma_\varphi = \Gamma_m/2$ [72], which is referred to as the quantum limit. In this limit, the only source of noise is the inevitable perturbation of the system due to the measurement, which is a purely quantum effect. Other noise sources will add to the decoherence such that the total dephasing and measurement rates must satisfy

$$\frac{\Gamma_\varphi}{\Gamma_m} \geq \frac{1}{2},$$

(5.70)

where the equality holds in the case of an optimal measurement in the above sense. It is thus natural to define the quantum efficiency

$$\eta = \frac{t_\varphi}{2t_m},$$

(5.71)

with $\eta \leq 1$. A measurement with $\eta = 1$ is called a quantum limited measurement. We note that the definition of measurement time $t_m = 1/\Gamma_m$ above is by no means universal. In Ref. [74], the measurement time is defined as the time for which the gaussian distributions can be distinguished

$$\mu_+(\bar{t}_m) - \mu_-(\bar{t}_m) = \sqrt{2}(\sigma_+(\bar{t}_m) + \sigma_-(\bar{t}_m)).$$

(5.72)

With the time dependence of the distributions for the homodyne measurement, this gives the measurement time $\bar{t}_m = 2/\Gamma_m = 2t_m$. With this definition, the quantum efficiency in [74] is correspondingly defined as

$$\tilde{\eta} = \frac{t_\varphi}{\bar{t}_m} \leq 1,$$

(5.73)
which agrees with the definition in Eq. (5.71). We stress that the differing factor of $1/2$ between the definitions of quantum efficiency in Eq. (5.71) and Eq. (5.73) is only due to the different definitions of measurement time. In Paper II and III we have chosen the former in accordance with standard quantum optics definitions [63] and to be consistent with Refs. [75] and [76].
Chapter 6

Summary and discussion

In this chapter, we briefly present the main results of the work contained in papers I to VI and relate the methods used in these papers to the theoretical tools that have been presented in this thesis.

6.1 Quantum limited read-out

The read-out of the SCB through its quantum capacitance, as was discussed semi-classically in Sec. 2.3, was analyzed in papers I to III. Here, the analysis of the system was performed using the Hamiltonian formalism for electrical circuits as was presented in Sec. 2.1. With this, the quantum capacitance was derived in a fully quantum mechanical framework which agreed with the prior semi-classical results. By calculating the measurement and dephasing rates this type of read-out was found to be quantum limited. Here, the quantum probe used for read-out is the microwave signal which is modeled as a coherent state of the electromagnetic field which was discussed in Sec. 3.3.2. In addition to this, unwanted degrees of freedom in the transmission line was modeled as thermally excited photons. With this, it was found that an oscillator, strongly coupled to the transmission line, shields against such excitations while simultaneously allowing for fast read-out.
6.2 Error-correction with super-conducting qubits

As discussed in Sec. 1.5 the implementation of error-correcting codes that are able to correct for an arbitrary error require too many qubits to be considered realizable in current superconducting qubit systems. It is therefore interesting to investigate what can be done with simpler codes that require less qubits to work. One such code is the three qubit phase flip code which can protect from single phase flips. As discussed in Sec. 4.3.2, this type of error is equivalent to dephasing, where the probability of error is related to the dephasing rate according to \( p = (1 - e^{-\Gamma \phi t})/2 \). Such a code can recover a quantum state perfectly in the case of dephasing errors but only in situations where the dephasing rate is low. However, for sufficiently large values of \( \Gamma \phi t \) it is clear that the probability of multiple errors cannot be ignored, and the effect of several qubits simultaneously getting phase-flipped must be accounted for. In paper IV, performance of this code in the presence of multiple errors is investigated where two specific schemes for qubit-qubit interaction realizable in superconducting systems are considered; one \( \sigma_z \sigma_z \)-coupling and one cavity mediated coupling. The first coupling obviously commute with the dephasing error operator which allows us to give analytical expressions for the code fidelity. This can not be done for the cavity mediated coupling where the performance of the code is investigated by numerically simulating the master equation given in Sec. 4.5. Considering errors during storing as well as processing, we calculate the maximum operation time allowed in order to still benefit from the code for both types of couplings and show that this limit can be reached with current technology.

6.3 High fidelity gates

One of the DiVincenzo criteria, as was listed in Sec. 1.2, states the necessity for good control of the qubit state in any physical realization of a quantum computer such that one can perform a universal set of gates. The first step towards good control is the ability to perform single-qubit gates with high fidelity. Apart from the need for excellent control of the experimental equipment such as signal generators, this also puts requirements on the decoherence times exhibited by the system. Any error process, such as relaxation or dephasing will prevent the gate from generating the desired output state and thus reduce the fidelity of the operation. In paper V, single qubit gates where experimentally implemented in a transmon qubit as presented
The fidelity was measured using three different metrics and low gate errors $\sim 1 - 2\%$ across all metrics were obtained. From a theoretical point of view, the performance of the gates was investigated numerically by simulating the master equation given in Sec. 4.6, including the effect of relaxation and dephasing. The resulting state generated by the gates was then calculated and compared to the expected state in the ideal case. Comparison with measured results showed that the limits on gate fidelity were primarily imposed by qubit decoherence.

6.4 Generation of entangled states in circuit QED

The fact that the measurement disturbs the system can be used in a constructive way to initialize a system in a desired state. The collapse of the wavefunction can thus be utilized to generate states that otherwise would require entangling gates to be produced. In paper VI, the generation of the state $|\psi\rangle = (|001\rangle + |110\rangle)/\sqrt{2}$ by the measurement of the operator $\sigma_1^z + \sigma_2^z - 2\sigma_3^z$ using a dispersive measurement in circuit QED was investigated. This operator has the eigenvalues $\{-4, -2, 0, 2, 4\}$ and, as described in Sec. 5.1, a measurement with the outcome 0 will project the state to the subspace belonging to that eigenvalue (in this case spanned by the states $|001\rangle$ and $|110\rangle$). A single qubit rotation on the third qubit then generates the GHZ-state which is interesting from the point of view of entanglement, as was discussed in Sec. 1.4 and 5.3. However, due to the finite measurement time, errors will have time to corrupt the state that is generated in the measurement and it is not certain that the post-measurement state is the desired one. This effect is investigated using a SME as the one derived in Sec. 5.4, where relaxation and dephasing during the measurement can be accounted for. Using this, it is shown that a this type of joint dispersive readout of several qubits can be utilized for the probabilistic production of high-fidelity GHZ states. The potential of the dispersive readout to demonstrate a violation of local hidden variable (LHV) theories is also discussed.
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Sarah, how easy is it not to write a thesis when every day ends with a smile and starts with some freshly made coffee! Your support and love means the world to me and I can’t imagine a life without you.
Appendix A

Derivation of $g_0$ and $\sum |f_j|^2$

Here we give the derivation of $g_0$ and $\sum |f_j|^2$ in the Markov approximation and show that they are $\propto \tau$. Given $f_j$ in Eq. (5.28) we calculate the integral explicitly giving

$$|f_j|^2 = 4|\kappa_j|^2 \sin^2 \left[ \frac{(\omega_j - \omega_0)\tau}{2} \right] \frac{(\omega_j - \omega_0)^2}{(\omega - \omega_0)^2}. \quad (A.1)$$

When summing over $|f_j|^2$, we make the assumption that the bath modes are so densely spaced in frequency, that they may be approximated by a continuum. Thus we write the sum as

$$\sum_j |f_j|^2 \to \int_0^\infty d\omega 4D(\omega)|\kappa(\omega)|^2 \frac{\sin^2 \left[ (\omega - \omega_0)\tau/2 \right]}{(\omega - \omega_0)^2}, \quad (A.2)$$

where $D(\omega)$ is the bath density of states. In Fig. A.1 we plot the function $\sin^2 \left[ (\omega - \omega_0)\tau/2 \right]/(\omega - \omega_0)^2$ as a function of the frequency difference $\omega - \omega_0$ for a fixed $\tau$. It is clear that it is only modes in the range $\omega - \omega_0 \sim 1/\tau$ that contribute to the integral in Eq. (A.2). If the time-scale $\tau$ is chosen so that the variation of $D(\omega)|\kappa(\omega)|^2$ is negligible over the range $[-2\pi/\tau, 2\pi/\tau]$, that is if

$$\frac{\partial}{\partial \omega} (D(\omega)|\kappa(\omega)|^2)_{\omega=\omega_0} \ll \tau, \quad (A.3)$$

is satisfied, we may take $D(\omega)|\kappa(\omega)|^2$ with its value at $\omega = \omega_0$ and take it outside the integral. The rest can be explicitly computed yielding

$$D(\omega_0)|\kappa(\omega_0)|^2 \int_0^\infty d\omega \frac{4\sin^2 \left[ (\omega - \omega_0)\tau/2 \right]}{(\omega - \omega_0)^2} = 2\pi D(\omega_0)|\kappa(\omega_0)|^2\tau \equiv \gamma_0\tau, \quad (A.4)$$

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where we have defined the qubit decay rate $\gamma_0$. Note that we could have obtained the same result using the standard limit
\[
\lim_{\alpha \to \infty} \frac{\sin^2(\alpha x)}{\alpha x} = \delta(x)
\] (A.5)
to write
\[
\lim_{\tau \to \infty} \frac{4 \sin^2 [(\omega - \omega_0)\tau/2]}{(\omega - \omega_0)^2} = 2\pi \tau \delta(\omega - \omega_0).
\] (A.6)
This would however not have highlighted on what time scale we need $\tau \to \infty$.

The calculation of $g_0$ is similar to that above with the exception that $g_0$ acquires an additional imaginary part
\[
g_0 = \int_0^\infty d\omega D(\omega)|\kappa(\omega)|^2 \left( \frac{2 \sin^2 [(\omega - \omega_0)\tau/2]}{(\omega - \omega_0)^2} + \frac{i \tau}{\omega_0 - \omega} - \frac{i \sin[(\omega_0 - \omega)\tau]}{(\omega_0 - \omega)^2} \right).
\] (A.7)
Again letting $\tau \to \infty$ as defined above, the first term can be well approximated by a $\delta$-function. It is clear that the imaginary part of the integrand is well-behaved for all values of $\omega$ and $\tau$ and equal to zero for $\omega = \omega_0$. As $\tau \to \infty$ the $\sin[(\omega_0 - \omega)\tau]$ is rapidly oscillating and we can approximate the imaginary part with its linear term. With this we write $g_0$ as
\[
g_0 = \frac{1}{2} \gamma_0 \tau + i\tau P \int_0^\infty d\omega \frac{D(\omega)|\kappa(\omega)|^2}{\omega - \omega_0},
\] (A.8)
where $\mathcal{P}$ denotes the Cauchy principal value (see Appendix B). The imaginary term shifts the energy levels of the unperturbed system and can be included in the unitary evolution. This does not alter the evolution qualitatively, as opposed to the influence of the real part which gives rise to non-unitary evolution of the system due to the measurement. Hence, the imaginary part has been ignored in Sec. 5.4.1.
Appendix B

Cauchy principal value

In this section, we briefly review the Cauchy Principal value. To this end, we consider the integral expression

\[ I = \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} ds f(\omega)e^{i\omega s}, \quad \text{(B.1)} \]

which can be rewritten as

\[ I = \lim_{\delta \to 0} \int d\omega \int_{0}^{\infty} ds f(\omega)e^{i(\omega+i\delta)s}, \quad \text{(B.2)} \]

where \( \delta \in \mathbb{R} \). The integral with respect to \( s \) can now be explicitly calculated yielding

\[ I = \lim_{\delta \to 0} \int d\omega f(\omega) \left( \frac{i\omega}{\omega^2 + \delta^2} + \frac{\delta}{\omega^2 + \delta^2} \right), \quad \text{(B.3)} \]

where we now consider the real and imaginary parts of this separately. Clearly, the real part is well behaved and goes to zero when \( \delta \to 0 \) for all \( \omega \neq 0 \). It is therefore enough to consider a small interval around the point \( \omega = 0 \) when calculating the integral

\[ \Re(I) = \lim_{\delta \to 0} \int_{-\epsilon}^{\epsilon} d\omega f(\omega) \frac{\delta}{\omega^2 + \delta^2} \approx \lim_{\delta \to 0} \int_{-\epsilon}^{\epsilon} d\omega \left( f(0) + f'(0)\omega \right) \frac{\delta}{\omega^2 + \delta^2} \]

\[ = \lim_{\delta \to 0} 2\delta f(0) \int_{0}^{\epsilon} d\omega \frac{\delta}{\omega^2 + \delta^2} = \lim_{\delta \to 0} 2\delta f(0) \arctan(\epsilon/\delta) = \pi f(0). \]

\[ \text{(B.4)} \]

This is valid for any \( f(\omega) \) which allow us to write, symbolically,

\[ \Re \left( \int_{0}^{\infty} ds e^{i\omega s} \right) = \pi \delta(\omega). \quad \text{(B.5)} \]
Appendix B

The imaginary part is similarly well behaved for all \( \omega \neq 0 \) and we consider again a small interval around \( \omega = 0 \)

\[
\lim_{\delta \to 0} \int_{-\epsilon}^{\epsilon} d\omega \frac{f(\omega)}{\omega^2 + \delta^2} \approx \lim_{\delta \to 0} \int_{-\epsilon}^{\epsilon} d\omega \left( f(0) + f'(0)\omega \right) \frac{\omega}{\omega^2 + \delta^2} = \lim_{\delta \to 0} 2\delta f'(0) \int_0^\epsilon d\omega \frac{\omega^2}{\omega^2 + \delta^2} = 2f'(0)\epsilon = 0, \quad (B.6)
\]

when \( \epsilon \to 0 \). We can thus exclude the point \( \omega = 0 \) when calculating the imaginary part of the integral in Eq. (B.1) and write this as

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} d\omega \frac{f(\omega)}{\omega} + \int_{\epsilon}^{\infty} d\omega \frac{f(\omega)}{\omega} \equiv \mathcal{P} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega}, \quad (B.7)
\]

which is the definition of the Cauchy principal value.
Appendix C

The Liouville equation in the diffusion limit

In this section, we derive Eq. (5.54) given in Sec. 5.4.2. The starting point is Eqs. (5.52) and (5.53)

\begin{equation}
T_0[\tilde{\rho}, t|\rho, t_0] = \left(1 - \tau \gamma_0 (\langle \sigma_+ \sigma_- \rangle + |\beta|^2)\right) \delta [\rho - i\tau G[\rho] - \tilde{\rho}],
\end{equation}

and

\begin{equation}
T_{1/2}[\tilde{\rho}, t|\rho, t_0] = \frac{1}{2}\tau \gamma_0 |\beta|^2 (1 \mp \epsilon C^\dagger)(1 \pm \epsilon C)\delta \left[\frac{(1 \mp \epsilon C)\rho (1 \mp \epsilon C)}{(1 \pm \epsilon C^\dagger)(1 \mp \epsilon C)} - \tilde{\rho}\right],
\end{equation}

where we have introduced the super-operator

\begin{equation}
G[\rho] = [V_{qb}, \rho] - \frac{i\gamma_0}{2}\{\sigma_+\sigma_- - \langle \sigma_+\sigma_- \rangle, \rho\},
\end{equation}

as short-hand. We are interested in the diffusive limit, that is the limit, where the jump-size goes to zero. To this end we write \(T_0, T_{1/2}\) and \(P[\tilde{\rho}, t_0]\) in terms of the jump \(\Delta \rho = \tilde{\rho} - \rho\) and the initial state \(\rho\). The assumption is now that the rates in Eqs. (C.1) and (C.2) vary smoothly with \(\rho\) but are sharply peaked around \(\Delta \rho = 0\). In addition, we assume that \(P[\rho, t_0]\) varies slowly on the scale \(\Delta \rho\). In this case we re-write the transition probability \(T[\tilde{\rho}, t|\rho, t_0] = f[\tilde{\rho} - \Delta \rho, \Delta \rho, t, t_0]\), and expand to second order in \(\Delta \rho\)

\begin{align*}
\int d\Delta \rho \ T[\tilde{\rho}, t|\rho, t_0] P[\rho, t_0] &= \int d\Delta \rho \ f[\tilde{\rho} - \Delta \rho, \Delta \rho, t, t_0] P[\tilde{\rho} - \Delta \rho, t_0] \\
&\approx \alpha[\tilde{\rho}, t_0] P[\tilde{\rho}, t_0] - \frac{\partial}{\partial \tilde{\rho}} \left(\beta[\tilde{\rho}, t, t_0] P[\tilde{\rho}, t_0]\right) \\
&+ \frac{1}{2} \frac{\partial^2}{\partial \tilde{\rho}^2} \left(\gamma[\tilde{\rho}, t, t_0] P[\tilde{\rho}, t_0]\right),
\end{align*}

(4)
where the pre-factors are given by

\[
\alpha[\tilde{\rho}, t, t_0] = \int d\Delta \rho \, f[\tilde{\rho}, \Delta \rho, t, t_0], \\
\beta[\tilde{\rho}, t, t_0] = \int d\Delta \rho \, f[\tilde{\rho}, \Delta \rho, t, t_0] \Delta \rho, \\
\gamma[\tilde{\rho}, t, t_0] = \int d\Delta \rho \, f[\tilde{\rho}, \Delta \rho, t, t_0] (\Delta \rho)^2.
\]  
(C.5)

Note that we have taken into account the strong dependence of \( f[\tilde{\rho} - \Delta \rho, \Delta \rho, t, t_0] \) on \( \Delta \rho \) by not expanding with respect to \( \Delta \rho \) in the second argument. On this form, we get from Eq. (C.1)

\[
\alpha_0[\tilde{\rho}, t, t_0] = 1 - \tau \gamma_0 \left( \langle \sigma_+ \sigma_- \rangle + |\beta|^2 \right), \\
\beta_0[\tilde{\rho}, t, t_0] = -i \tau G[\tilde{\rho}] + \mathcal{O}(\tau^2),
\]

where \( \gamma[\tilde{\rho}, t, t_0] \propto \tau^2 \) and has been omitted. To get Eq. (C.2) on the right form, we first expanded it to second order in the small parameter \( \epsilon \) giving the \( \delta \)-function of the \( T_2 \)-term in Eq. (C.2)

\[
\delta \left[ \frac{(1 + \epsilon C) \rho (1 + \epsilon C^\dagger)}{\langle (1 + \epsilon C^\dagger)(1 + \epsilon C) \rangle} - \hat{\rho} \right] \approx \delta \left[ \rho - \hat{\rho} + \epsilon \mathcal{M}[\rho] + \epsilon^2 \mathcal{N}[\rho] \right],
\]

(C.7)

where the super-operators

\[
\mathcal{M}[\rho] = \rho C^\dagger + C \rho - \langle C + C^\dagger \rangle \rho, \\
\mathcal{N}[\rho] = C \rho C^\dagger + \langle C + C^\dagger \rangle^2 \rho - \langle C^\dagger C \rangle \rho - \langle C + C^\dagger \rangle (\rho C^\dagger + C \rho),
\]

(C.8)

have been introduced as short-hand. The pre-factors in Eq. (C.5) can then be calculated for the \( T_{1/2} \)-terms giving

\[
\alpha_{1/2}[\tilde{\rho}, t, t_0] = \frac{1}{2} \tau \gamma_0 |\beta|^2 \left( 1 + \epsilon \langle C + C^\dagger \rangle + \epsilon^2 \langle C^\dagger C \rangle \right), \\
\beta_{1/2}[\tilde{\rho}, t, t_0] = \frac{1}{2} \tau \gamma_0 |\beta|^2 \left( \epsilon \mathcal{M}[\tilde{\rho}] + \epsilon^2 (\mathcal{N}[\tilde{\rho}] + \langle C + C^\dagger \rangle \mathcal{M}[\tilde{\rho}] \right), \\
\gamma_{1/2}[\tilde{\rho}, t, t_0] = \frac{1}{2} \tau \gamma_0 |\beta|^2 \epsilon^2 \mathcal{M}^2[\tilde{\rho}],
\]

(C.9)

where we have kept terms up to second order in \( \epsilon \). It is clear that, when added, the terms that are linear in \( \epsilon \) cancel. We now add the \( \alpha, \beta \) and \( \gamma \)
from Eqs. (C.6) and (C.9), yielding the Liouville equation in the diffusion limit

$$\frac{\partial}{\partial t} P[\hat{\rho}, t_0] = -\frac{\partial}{\partial \hat{\rho}} \left( \left( -i[V_{qb}, \hat{\rho}] + \gamma_0 D[\sigma_{-}] \hat{\rho} \right) P[\hat{\rho}, t_0] \right) + \frac{1}{2} \gamma_0 \frac{\partial^2}{\partial \hat{\rho}^2} \left( \mathcal{M}^2 P[\hat{\rho}, t_0] \right).$$

(C.10)

This is Eq. (5.54) in Sec. 5.4.2.
Bibliography


[34] J. S. Bell, Physics, 1, 195 (1964)


[57] Y. Mahklin, G. Schön and A. Shnirman, Rev. of Mod. Phys. 73 357 (2001)


[67] Christopher C. Gerry and Peter L. Knight, Introductory Quantum Optics, Cambridge University Press, 2005


[70] Fima C. Klebaner, Introduction to stochastic calculus with applications, Imperial college press, 2005


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